INDEX

MATHEMATICAL PHYSICS

CHAPTER NO.	CHAPTER NAME	PAGE NO.
1	Vector Calculus	1
2	Matrices	22
3	Differential Equation	39
4	Special Functions	61
5	Fourier Series	71
6	Laplace Transforms	83
7	Fourier Transform	100
8	Complex Analysis	109
9	Numericals Analysis	142
10	Probability	154
11	Group Theory	168
12	Tensor	172

Chapter 1

Vector Calculus

COORDINATE SYSTEMS AND TRANSFORMATION.

A **coordinate system** is a method for identifying the location of a point

There are three types of coordinate systems:-

i. Cartesian coordinate system
ii. Cylindrical coordinate system
iii. Spherical coordinate system

Cartesian Coordinate System

A point P can be represented as (x, y, z).The ranges of the coordinate variables x, y, and z are

$-\infty < x < \infty$

$-\infty < y < \infty$

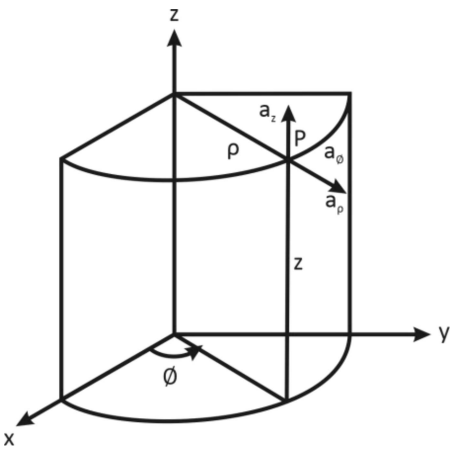
$-\infty < z < \infty$

A vector A in Cartesian (otherwise known as rectangular) coordinates can be written as (A_x, A_y, A_z) Or

$$\boldsymbol{A} = A_x a_x + A_y a_y + A_z a_z$$

Where a_x, a_y, and a_z are unit vectors along the x, y and z directions.

CIRCULAR CYLINDRICAL COORDINATES$(\boldsymbol{\rho}, \boldsymbol{\phi}, \boldsymbol{z})$



Azimuthally angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$$0 \leq \rho < \infty$$

$$0 \leq \phi < 2\pi$$

$$-\infty < z < \infty$$

A vector **A** in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z)

$$\boldsymbol{A} = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$$

Where a_ρ, a_θ and a_ϕ are unit vectors along the $r-, \theta -$, andϕ-directions.

The magnitude of A is

$$|A| = \left(A_\rho^2 + A_\theta^2 + A_\theta^2\right)^{\frac{1}{2}}$$

The unit vectors a_ρ, a_θ and a_ϕ are mutually orthogonal; a_r being directed along the radius or in the direction of increasing r, a_θ in the direction of increasingθ, and a_ϕ in the direction of increasingϕ. Thus,

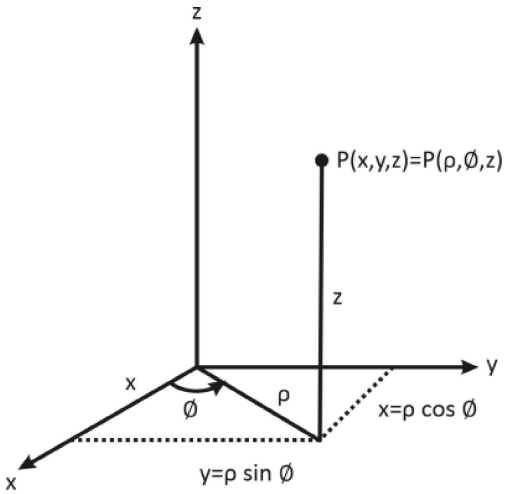
$$a_\rho . a_\rho = a_\theta . a_\theta = a_\phi . a_\phi = 1$$
$$a_\rho . a_\theta = a_\theta . a_\theta = a_\phi . a_\rho = 0$$
$$a_\rho \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_\rho$$
$$a_\phi \times a_\rho = a_\theta$$

The relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from Figure 2.2 as

$$\rho = \sqrt{x^2 + y^2}\,, \phi = \tan^{-1}\frac{y}{x}, z = z$$

Or

$$x = \rho\cos\phi, y = \rho\sin\phi, z = z$$



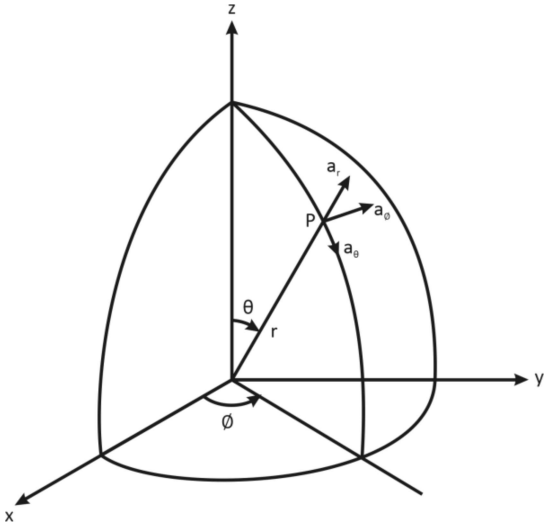
SPHERICAL COORDINATES (r, θ, ϕ)

A point P can be represented as (r, θ, ϕ)

r is the distance from the origin to point P or the radius of a sphere,

θ is the angle between the z-axis and the position vector of P;

ϕ is measured from the x-axis,



According to these definitions, the ranges of the variables are
$0 \leq r < \infty; 0 \leq \theta \leq \pi; 0 \leq \phi < 2\pi$

A vector A in spherical coordinates may be written as (A_r, A_θ, A_ϕ)

$$A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

Where a_r, a_θ and a_ϕ are unit vectors along the r, θ, and ϕ directions.
The magnitude of A is

$$|A| = (A_r^2 + A_\theta^2 + A_\theta^2)^{\frac{1}{2}}$$

The unit vectors a_r, a_θ and a_ϕ are mutually orthogonal; a_r being directed along the radius or in the direction of increasing r, a_θ in the direction of increasing θ, and a_ϕ in the direction of increasing ϕ. Thus,

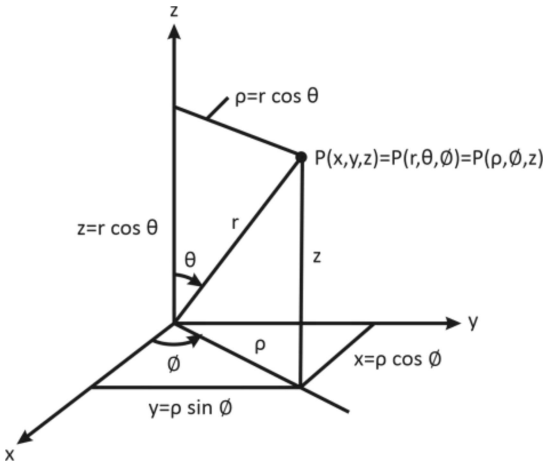
$$a_r . a_r = a_\theta . a_\theta = a_\phi . a_\phi = 1$$
$$a_r . a_\theta = a_\theta . a_\theta = a_\phi . a_r = 0$$
$$a_r \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_r$$
$$a_\phi \times a_r = a_\theta$$

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From Figure 2.5 it is easy to notice that

$r = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}, \phi = \tan^{-1} \frac{y}{x}$

Or

$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta$



Vector Calculus:-

Vector calculus, also known as vector analysis, deals with differentiation and integration of the vector field.

Physical Quantity:-

Those quantities which can be measured by physical Instruments called physical quantity.

Eg.→Mass. Length, temperature, time, etc.

Physical quantities are two types s

- →scalar quantity
- →vector quantity

Scalar Quantity:-

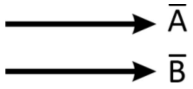
They have only magnitude.

Eg. mass; time; length, etc.

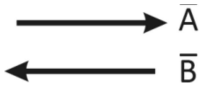
Vector Quantity:- They have magnitude as well as direction.
Eg. force, momentum, torque.

Some Important Vectors:-

Equal Vectors:- If two vectors have equal magnitude and same direction.
i.e.$\overline{A} = \overline{B}$



Negative Vectors:- If two vectors have equal magnitude but opposite in direction.
i.e.$\overline{A} = -\overline{B}$



Collinear Vectors:-Vectors having equal or unequal magnitudes but drawn in the same direction.



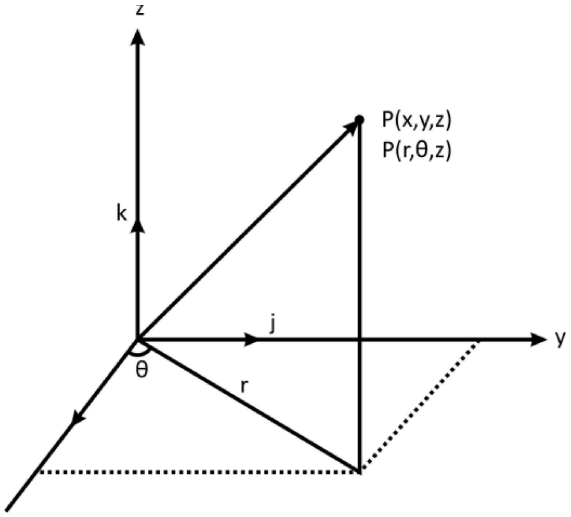
Unit Vector:- A vector of unit magnitude and whose direction in same as that of the given vector is called unit vector.

$$\overline{A} = |\overline{A}|\hat{A}$$

$\because |A| = 1$

$$\hat{A} = \frac{\overline{A}}{|A|}$$

Vectors expressed in terms of unit vectors



$$\overline{A} = x\hat{i} + y\hat{j} + z\hat{k}$$

$$|\overline{A}| = \sqrt{x^2 + y^2 + z^2}$$

$$\hat{A} = \frac{\overline{A}}{|\overline{A}|}$$

$$\Rightarrow \frac{x\hat{i} + y\hat{j} + z\hat{k}}{\sqrt{x^2 + y^2 + z}}$$

Multiplication of Vectors:-

- Scalar Product
- Dot Product
- Cross Product

Scalar Product:- Product of a scalar m to a $\overline{A}$

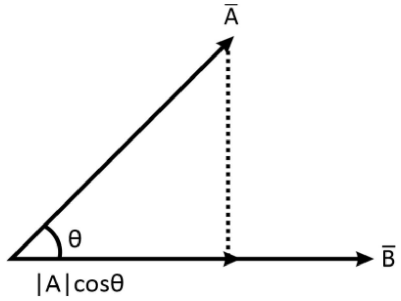
$$\overline{\boldsymbol{R}} = \boldsymbol{m}\overline{\boldsymbol{A}}$$
$$\overline{\boldsymbol{R}} = \boldsymbol{mA_x\hat{\imath} + m\,A_y\hat{\jmath} + mA_z\hat{k}}$$

Dot Product:-

The dot product of two vectors $\overline{A}$ and $\overline{B}$ in expressed with a "dot" between the two vectors as:-

$$\overline{\boldsymbol{A}} \cdot \overline{\boldsymbol{B}} = \boldsymbol{|A||B|\cos\theta}$$

Dot product of two vectors will be a scalar they have only magnitude.



Example: Find the scalar product of$\overline{A} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ and $\overline{B} = 4\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$?

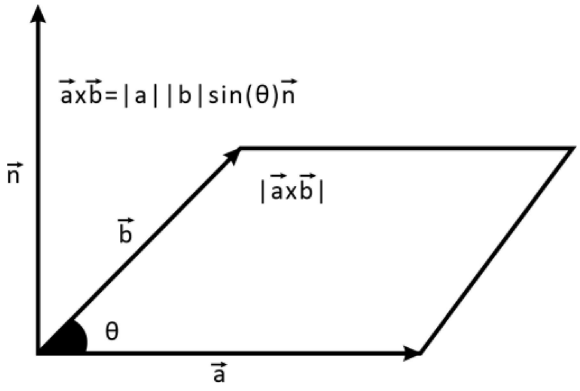
Solution:

$\overline{A} \cdot \overline{B} = \left(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right) \cdot (4\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$

$= 8 + 6 + 6 \Rightarrow 20$ unit.

Cross Product:- If $\overline{A}$ and $\overline{B}$ are two vectors, then the result of the cross product of these two vectors $\left(\overline{A} \times \overline{B}\right)$ in perpendicular to the both the vectors and normal to plane that contains both the vectors.

$$\overline{\boldsymbol{A}} \times \overline{\boldsymbol{B}} = \boldsymbol{|A||B|\sin\theta\,\hat{n}}$$



- The angle between $\vec{a}$ and $\vec{c}$ is always 90.i.e., $\vec{a}$ and $\vec{c}$ are orthogonal vectors.
- The angle between $\vec{b}$ and $\vec{c}$ is always 90 .i.e., $\vec{b}$ and$\vec{c}$are orthogonal vectors.
- We can position $\vec{a}$ and $\vec{b}$ parallel to eachother or at an angle of O°, making the resultant vector a zero vector.
- To get the greatest magnitude, the original vectors must be perpendicular (angle of 90°) so that the cross product of the two vectors will be maximum.

Cross Product Properties:-

- The length of the cross product of two vectors :

$$\vec{a} \times \vec{b} = |a||b| \sin(\theta).$$

- Anti-commutative property:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

- Distributive property:

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + (\vec{a} \times \vec{c})$$

- Cross product of the zero vector:

$$\vec{a} \times \vec{O} = \vec{O}$$

- Cross product of the vector with itself:

$$\vec{a} \times \vec{a} = \vec{O}$$

- Multiplied by a scalar quantity:

$$\vec{c}(\vec{a} \times \vec{b}) = c\,\vec{a} \times \vec{b} = \vec{a} \times \vec{c}\vec{b}$$

- The cross product of the unit vectors:

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$
$$i \times j = k;\ j \times k = i;\ \dot{k} \times i = j$$

Triple Cross Product

The cross product of a vector with the cross product of the other two vectors is the triple cross product of the vectors. The resultant of the triple cross product is a vector. The resultant of the triple cross vector lies in the plane of the given three vectors.

If **a**, **b**, and **c** are the vectors, then the vector triple product of these vectors will be of the form:

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

Note:-

$$\overline{a} \times (\overline{b} \times \overline{c}) + \overline{b} \times (\overline{c} \times \overline{a}) + \overline{c} \times (\overline{a} \times \overline{b}) = 0$$

Example 1:- The expression $\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)$ is a

(a) unit vector

(b) Null vector

(c) Vector of magnitude

(d) Scalar

Solution:-

Given that,

$$\overline{A} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$$

$$|\overline{A}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \Rightarrow \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$|\overline{A}| = 1$ (unit vector)

Option **a** is correct

Example 2:-

If for two vectors $\overline{a}$ and $\overline{b}$

$|\overline{a} \times \overline{b}| = |\overline{a} - \overline{b}|$

Find the angle between $\overline{a}$ and $\overline{b}$

Solution:-

$$|\bar{a}+\bar{b}| = |\bar{a}-\bar{b}|$$
$$|a+b|^2 = |\bar{a}+\bar{b}|^2$$
$$(\bar{a}+\bar{b}).(\bar{a}+\bar{b}) = (\bar{a}-\bar{b}).(\bar{a}-\bar{b})$$
$$\bar{a}^2+\bar{b}^2+2\bar{a}.\bar{b} = \bar{a}^2+\bar{b}^2-2\bar{a}.\bar{b}$$
Or $\bar{a}\cdot\bar{b}=0$

Thus angle between **a** and **b** is **90**◦.

Example 3:-

If $\bar{a},\bar{b},\bar{c}$ be three unit vectors such that $\bar{a}\times(\bar{b}\times\bar{c}) = \frac{1}{2}\bar{b}$, find the angles which $\bar{a}$ makes with $\bar{b}$ and $\bar{c}$ and $\bar{b}$ and $\bar{c}$ being non parallel.

Solution:-

$$\bar{a}\times(\bar{b}\times\bar{c}) = \frac{1}{2}\bar{b}$$
$$(\bar{a}.\bar{c})\bar{b}-(\bar{a}.\bar{b})\bar{c} = \frac{1}{2}\bar{b}$$
$$\bar{a}.\bar{c} = \tfrac{1}{2} \text{ and } \bar{a}.\bar{b} = 0$$

$\Rightarrow \bar{a}\perp\bar{b}$

Now $\bar{a}.\bar{c} = \frac{1}{2}$

$$|a||c|\cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2}$$
$$\theta = \cos^{-1}\frac{1}{2}; \theta = \frac{\pi}{3}$$

Gradient:- Gradient of a scalar field at a point represent the maximum rate of change of scalar field with respect to position and is always directed along the normal to the field surface at that point & its defined as

$$grad\ \phi = \bar{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{\imath}+\frac{\partial}{\partial y}\hat{\jmath}+\frac{\partial}{\partial z}\hat{k}\right)\phi$$
$$\bar{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\imath}+\frac{\partial\phi}{\partial y}\hat{\jmath}+\frac{\partial\phi}{\partial z}\hat{k}$$

It can be written as –

$$d\phi = \left(\hat{\imath}\frac{\partial\phi}{\partial x}+\hat{\jmath}\frac{\partial\phi}{\partial y}+\hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{\imath}dx+\hat{\jmath}dy+\hat{k}dz\right)$$
$$= (\nabla\phi).dr$$
$$d\phi = |\nabla\phi||dr|\cos\theta$$

If $d\bar{r}$ and $\nabla\phi$ are in the same direction,

Then $\theta = 0$ thus $\cos\theta = 1$

$$d\phi = |\nabla\phi||dr|$$

Normal derivative:-

If $\phi(x,y,z) = k$ respresent a family of surfaces for different values of the constant k.

Then $d\phi = 0$

So, $\nabla\phi.dr = 0$

Thus $\nabla\phi$ is a vector normal to the surface $\phi(x,y,z) = k$.

Directional derivative:-

If $d\phi \neq 0$

$$\nabla\phi.dr \neq 0$$

Example:-

The temperature at any point in space is given by $T = xy + yz + zx$.

Determine the derivative of T in the direction of the vector $3\hat{\imath} - 4\hat{k}$. At the point (1, 1, 1)

Solution:- Given that

$T = xy + yz + zx$

$\nabla T = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)T$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \hat{\imath}(y + z) + \hat{\jmath}(x + z) + \hat{k}(x + y)$

Directional derivative at $(1,1,1)$

$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Directional derivative in the direction $(3\hat{\imath} - 4\hat{\jmath})$

$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.\left(\frac{3\hat{\imath} - 4\hat{\jmath}}{\sqrt{9 + 16}}\right)$

$= \frac{6 - 8}{5} \Rightarrow -\frac{2}{5}$

Divergence of a Vector Field:-

The divergence of a vector function $\overline{A}$ is denoted by $div\ \overline{A}$

$$\overline{\nabla}.\overline{A} = \left(\hat{\imath}\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

Note:- $div\ \boldsymbol{A}$ is a scalar quantity.

If $u = x^2 + y^2 + z^2$ and $\overline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then find $div\ (4\overline{r})$ in term of 4.

$$div\ (u\overline{r}) = \overline{\nabla}.(u\overline{r})$$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[x^2 + y^2 + z^2]\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2)$

$= 5x^2 + 5y^2 + 5z^2$

$= 5(x^2 + y^2 + z^2)$

$= 5u$

Curl of a Vector Field:- The curl of a vector point function $\overline{A}$ is defined as known

$$\text{Curl}\ \overline{A} = \overline{\nabla} \times \overline{A}$$

$$\overline{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$$

$$\overline{\nabla} \times \overline{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$$= \hat{\imath}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$$

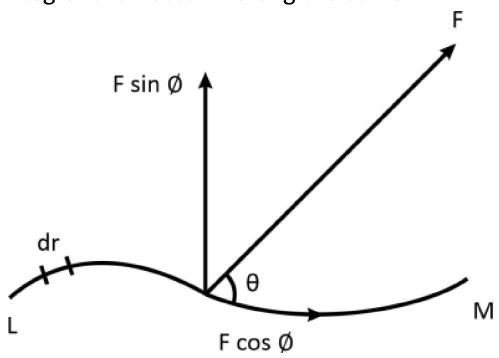
Example:- Calculate the curl of the vector $xyz\ \hat{\imath} + 3x^2y\hat{\jmath} + (xz^2 - y^2z)\hat{k}$

$$curl\ \overline{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\overline{\nabla} \times \overline{A} = -2yz\hat{\imath} + (xy - z^2)\hat{\jmath} + (6xy - xz)\hat{k}$$

Line Integral:-

Let $f(xyz)$ be the vector field "The line integral of a vector $\overline{F}$ along the curve LM"



$$I_\ell = \int_L^M (F\cos\theta)dr$$

$$I_\ell = \int_L^M \overline{F}.\overline{dr}$$

- If $\theta = 0^\circ$, along the curve LM ; $\cos\theta = 1$
  $I_L = \int_L^M \overline{F}.\overline{dr}$ (Maximum)
- If $\theta = 180^\circ$, opposite the curve LM ; $\cos 180^\circ = -1$
  $I_L = \int_L^M \overline{F}.d\overline{r}$ (Minimum).
- If $\theta = 90^\circ$, Perpendicular to the curve LM ; $\cos 90^\circ = 0$
  $I_L = \int_L^M \overline{F}.d\overline{r}$ (Zero).

Example: - Compute the line integral of **F** along a path that runs counter clockwisearound the unit circle.
Answer:- We parameterize C by $x = \cos\theta$, $y = \sin\theta$ with $0 \le \theta < 2\pi$.
Then $dx = -\sin\theta\, d\theta$, $dy = \cos\theta\ d\theta$, and:
$\int \boldsymbol{F}.dr = \int_{\theta=0}^{\theta=2\pi} -\sin\theta\, dx + \cos\theta\, dy = \int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\, d\theta = 2\pi.$

Example:- Compute the line integral along the path that goes from (0,0) to (1, 1) by first going along the x-axis to (1, 0) and then going up one unit to (1, 1).
Answer:- To Compute $\int F \cdot dr$ we break the curve into two pieces, then add the line integrals along each piece.
First, fix $y = 0$ (so $dy = 0$) and let x range from 0 to 1.

$$\int_{x=0}^{x=1} F \cdot dr = \int_{x=0}^{x=1} -ydx + xdy = \int_0^1 0\, dx = 0.$$

Next, fix $x = 1$ (so $dx = 0$) and let y range from 0 to 1:

$$\int_{y=0}^{y=1} F \cdot dx = \int_{y=0}^{y=1} -ydx + 1\, dy = 1.$$

We conclude that $\int \boldsymbol{F}.dr = 1.$

Surface Integrals:-

Let $F(x, y, z)$ be the vectors