P R E F A C E

This book on **'Applied Mathematics'** has been specially written as per latest syllabus of **'Applied Mathematics'** in **CSIR NET MATHEMATICAL SCIENCE EXAM**.

In addition this book will prove very useful for students preparing for various national level competitive examinations such as SET, GATE, TIFR, NBHM, IIT, JNU, DU, BHU, HCU, Deemed, Private, State, and Central University PhD Entrance Exams.

The author possesses a very long and rich experience of teaching mathematics and has first-hand experience of the problems and difficulties that students generally face.

The salient features of this book are:

1. The matter has been presented in a simple and lucid language so that student themselves shall be able to understand the concepts given and solution of the problems.
2. Each chapter open with necessary definitions and proofs of a standard results and theorems. These in turn are followed by solved examples which has been classified in various types and methods. This classification will help the students to revise the subject matter at the time of examination without losing any confidence.
3. Care has been taken not to omit important steps so that student can understand everything by themselves. Furthermore, a set of unsolved assignments is given in end of each chapter to instill confidence in the students.
4. The presentation in book is uncluttered, with plenty of clear outline diagrams.

In view of this special features, it is sincerely hoped that this book will surely serve its purpose.

All valuable suggestions for further improvement of the book will be highly appreciated.

A C K N O W L E D G E M E N T

Writing this book **"Applied Mathematics"** has been an incredible journey. All the chapters have taken a final shape after endless inputs of time and effort. Though many colleagues assisted me in compiling this book, I first of all would like to thank all students at Institute for advanced Studies (IFAS), who have helped me to learn and implement the concepts and ideas of Applied Mathematics in its simplest form.

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I N D E X

SECTION – I: ORDINARY DIFFERENTIAL EQUATIONS		
Chapter No.	***Chapter Name***	***Page No.***
1	**BASICS OF DIFFERENTIAL EQUATIONS**	***1***
	1.1 Introduction	***1***
	1.2 Definition of Differential Equation	***1***
	1.3 Classification of Differential Equations	***2***
	1.4 Solution of a Differential Equation	***2***
	1.5 Formation of Ordinary Differential Equation	***2***
2	**METHODS OF SOLVING FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS**	***4***
	2.1 Variable Separable Form	***4***
	2.2 Homogeneous Equations	***5***
	2.3 Exact Differential Equations	***6***
	2.4 Linear Differential Equations & Equations Reducible to Linear Differential Equations	***8***
	2.5 Bernoulli's Differential Equations	***9***
3	**HIGHER ORDER ORDINARY DIFFERENTIAL EQUATIONS**	***12***
	3.1 Preliminary Theory of Linear Differential Equations	***12***
	3.2 Wronskian Theory	***13***
	3.3 Higher Order Linear Differential Equations with Constant Coefficients	***17***
	3.4 Cauchy Euler Linear Differential Equations	***23***
	3.5 Variation of Parameters Method	***24***
	3.6 Reduction of Order Method	***26***
4	**BOUNDARY VALUE PROBLEMS**	***28***
	4.1 Boundary Value Problem	***28***
	4.2 Analysis of Zeros of Solutions.	***28***
	4.3 Sturm-Liouville's Problem	***29***
	4.4 Properties of Sturm-Liouville's Problem	***31***
5	**SYSTEMS OF FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS**	***33***
	5.1 Solving System by Elimination	***33***

	5.2 Homogeneous Linear System	***35***
	5.3 Autonomous System	***38***
6	**EXISTENCE & UNIQUENESS OF SOLUTIONS TO I.V.P. OF FIRST ORDER**	***40***
	6.1 Fundamental theorem of initial value problems	***40***
	6.2 Picard's successive approximations	***40***
	6.3 Lipschitz function	***41***
	6.4 Picard's Existence Theorem	***41***
SECTION – II: PARTIAL DIFFERENTIAL EQUATIONS		
Chapter No.	***Chapter Name***	***Page No.***
1	**BASICS OF PARTIAL DIFFERENTIAL EQUATIONS**	***44***
	1.1 Definition of Partial Differential Equation	***44***
	1.2 Order and Degree of Partial Differential Equation	***44***
	1.3 Classification of First Order Partial Differential Equation	***44***
	1.4 Formation of Partial Differential Equation by Elimination of an Arbitrary Constants	***45***
	1.5 Formation of Partial Differential Equation by Elimination of Arbitrary Function.	***46***
2	**FIRST ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS**	***49***
	2.1 Lagrange's Equation	***49***
	2.2 Integral Surfaces Passing through a given Curve (The Cauchy Problem)	***52***
	2.3 Uniqueness of Cauchy Problem	***55***
	2.4 Surfaces Orthogonal To A Given System Of Surfaces	***56***
3	**FIRST ORDER NON-LINEAR PARTIAL DIFFERENTIAL EQUATIONS**	***60***
	3.1 Types of Solutions	***60***
	3.2 Compatible System of Partial Differential Equations	***61***
	3.3 Charpit's Method for Solving First Order Non-linear Partial Differential Equations	***62***
	3.4 Method of Characteristics for Solving Non-linear Partial Differential Equations	***66***
4	**HIGHER ORDER LINEAR PARTIAL DIFFERENTIAL EQUATIONS WITH CONSTANT COEFFICIENTS & VARIABLE COEFFICIENTS**	***69***
	4.1 Introduction	***69***
	4.2 Complementary Functions	***69***
	4.3 Particular Integrals	***71***

	4.4 Non-homogeneous Partial Differential Equations (Reducible to Linear Factors)	***73***
5	**SECOND ORDER PARTIAL DIFFERENTIAL EQUATIONS**	***75***
	5.1 Second Order Partial Differential Equation & Classification	***75***
	5.2 Reduction To The Canonical Form	***75***
	5.3 Characteristic Roots and Characteristic Curves For Second Order Partial Differential Equations	***76***
6	**SEPERATION OF VARIABLES**	***79***
	6.1 Separation of Variables	***79***
	6.2 Heat Equation	***80***
	6.3 Wave Equation	***83***
	6.4 Laplace Equation	***85***
SECTION – III: INTEGRAL EQUATIONS		
Chapter No.	***Chapter Name***	***Page No.***
1	**PRELIMINARY CONCEPTS**	***89***
	1.1 Introduction	***89***
	1.2 Classification of An Integral Equations	***89***
	1.3 Solution of an Integral Equation	***90***
	1.4 Leibnitz Rule for Differentiation Under Integral Sign	***91***
	1.5 Converting Multiple Integrals Onto Single Integral	***91***
2	**CONVERSION OF ORDINARY DIFFERENTIAL EQUATIONS INTO INTEGRAL EQUATIONS**	***93***
	2.1 Introduction	***93***
	2.2 Method of Converting an Initial Value Problem Into A Volterra Integral Equation	***93***
	2.3 Conversion of an Integral Equation Into Differential Equation.	***94***
	2.4 Method of Converting A Boundary Value Problem Into A Fredholm Integral Equation	***94***
3	**HOMOGENEOUS FREDHOLM INTEGRAL EQUATIONS OF THE SECOND KIND WITH SEPERABLE OR DEGENERATE KERNELS**	***98***
	3.1 Characteristic Values/ Characteristic Numbers/Eigen-Values and Characteristic Functions (Or Eigen-Functions)	***98***
	3.2 Solution of Homogeneous Fredholm Integral Equation of The Second Kind with Separable Kernel	***98***
	3.3 Solution of Fredholm Integral Equation of The Second Kind with Separable Kernel	***102***

4	**METHOD OF SUCCESSIVE APPROXIMATIONS**	***105***
	4.1 Introduction	***105***
	4.2 Iterated Kernels or Iterated Functions	***105***
	4.3 Some Important Results	***105***
	4.4 Solution of Volterra Integral Equation of The Second Kind By Successive Approximations	***105***
	4.5 Solution of Fredholm Integral Equation with The Help of Resolvent Kernel	***106***
SECTION – IV: CALCULUS OF VARIATION		
Chapter No.	***Chapter Name***	***Page No.***
1	**CALCULUS OF VARIATION**	***110***
	1.1 Calculus of Variation	***110***
	1.2 Difference Between 'Function' & 'Functional'	***110***
	1.3 Simple Variational Problem	***110***
2	**ISOPERIMETRIC PROBLEM**	***119***
	2.1 Isoperimetric Problem	***119***
	2.2 Important Results From Calculus	***120***
3	**VARIATIONAL PROBLEM**	***123***
	3.1 Variational Problem For Functional Involving Several Dependent Variables	***123***
	3.2 Functions Involving Higher Derivatives	***124***
	3.3 Variation Problems Involving More Independent Variables:	***126***
4	**VARIATIONAL PROBLEM WITH MOVING BOUNDARIES**	***127***
	4.1 Variational Problems with Moving Boundaries:	***127***
	4.2 Boundary Value Problem & Rayleigh-Ritz Method	***129***
	4.3 Weierstrass Function	***130***
	4.4 Legendre Condition:	***131***
SECTION – V: NUMERICAL ANALYSIS		
Chapter No.	***Chapter Name***	***Page No.***
1	**SOLUTION OF ALGEBRAIC EQUATIONS**	***133***
	1.1 Some Important Results	***133***
	1.2 Errors	***133***
	1.3 Solution of Algebraic Equations	***136***

	1.4 Bisection Method	***136***
	1.5 Method of False Position	***137***
	1.6 Iteration Method	***138***
	1.7 Secant Method	***145***
2	**SOLUTION OF SYSTEM OF LINEAR ALGEBRAIC EQUATIONS**	***149***
	2.1 Triangular Matrices	***149***
	2.2 Gauss Elimination	***150***
	2.3 Gauss Jordan Method	***151***
	2.4 Gauss Jacobi Iteration Method	***152***
	2.5 Gauss- Seidel Iteration Method	***153***
3	**INTERPOLATION**	***158***
	3.1 Finite Differences	***158***
	3.2 Symbolic Relations	***159***
	3.3 Newton's Formula For Interpolation	***160***
	3.4 Lagrange Interpolation Formula	***161***
	3.5 Hermite's Interpolation Formula	***163***
	3.6 Spline Interpolation	***164***
	3.7 Divided Difference	***165***
4	**NUMERICAL INTEGRATION & DIFFERENTIATION**	***168***
	4.1 Numerical Integration	***168***
	4.2 Numerical Differentiation	***173***

SECTION – I

ORDINARY DIFFERENTIAL EQUATIONS

CHAPTER 1

BASICS OF DIFFERENTIAL EQUATIONS

1.1 Introduction:

The words differential and equations certainly suggest solving some kind of equation that contains derivatives $y', y'', \dots, y^n$ Analogous to a course in algebra and trigonometry, in which a good amount of time is spent solving equations such as $x^2 + 5x + 2 = 0$ for the unknown number x, in this course one of our tasks will be to solve differential equations such as $y'' + 5y' + 2y = 0$ for an unknown function $y(x)$.

Practical questions such as how fast does a Corona Virus spread? How fast does a population change? Involve rates of change or derivatives. And probably such Mathematical experiments leads to a study of Differential Equations.
Firstly let's understand what **Differential Equation** is.

1.2 Definition: An equation containing the derivatives of one or more dependent variables, with respect to one or more independent variables, is said to be a differential equation (DE).
Example:
(1) $\frac{dy}{dx} + y = co$
(2) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$
(3) $\frac{dx}{dt} + y = 1, \frac{dy}{dt} + x = 2$
(4) $\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^3 = xy$

1.3 Let's classify differential equations by type, order and linearity,

1.3.1 Classification By Type: If an equation contains only ordinary derivatives of one or more dependent variables with respect to a single independent variable it is said to be an **ordinary differential equation (ODE).**

For example, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = y\frac{d^3y}{dx^3}$

An equation involving partial derivatives of one or more dependent variables of two or more independent variables is called a **Partial differential equation (PDE).**
Examples:
(1) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = x$,

(2) $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = xy,$

1.3.2 Classification By Order of Differential Equation:
The order of a differential equation (either ODE or PDE) is the order of the highest derivative in the equation.

For example,

1) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = y\frac{d^3y}{dx^3}$

is a third-order ordinary differential equation

2) $\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = xy$

is a first-order ordinary differential equation

1.3.3 Classification By Linearity:
An nth-order ordinary differential equation $F(x, y, y', y'', \dots, y^n) = 0$ is said to be linear if F is linear in $y, y', y'', \dots, y^n$ This means that an nth-order ODE $F(x, y, y', y'', \dots, y^n) = 0$ is linear when is of the form $a_0(x)\frac{d^ny}{dx^n} + a_1(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_{n-1}(x)\frac{dy}{dx} + a_n(x)y = g(x)$……*

Example, $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = x$ is linear differential equation.

In the additive combination on the left-hand side of equation * we see that the two properties of a linear ODE are as follows:

- The dependent variable y and all its derivatives $y, y', y'', \dots, y^n$ are of the first degree, that is, the power of each term involving y is 1.
- The coefficients $a_0, a_1, \dots, a_n$ of $y, y', y'', \dots, y^n$depend at most on the independent variable x

A nonlinear ordinary differential equation is simply one that is not linear. Nonlinear functions of the dependent variable or its derivatives, such as $\sin y$ or, cannot appear in a linear equation. Therefore
Example:

1) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} = y\frac{d^3y}{dx^3}$

2) $\frac{d^2y}{dx^2} + \sin y = 0$

are examples of nonlinear ordinary differential equations.

1.4 Solution Of An Ode:

Any function φ, defined on an interval I and possessing at least n derivatives that are continuous on I, which when substituted into an nth-order ordinary differential equation reduces the equation to an identity, is said to be a solution of the equation on the interval.

In other words, a solution of an nth-order ordinary differential equation $F(x, y, y', y'', \dots, y^n) = 0$ is a function φ that possesses at least n derivatives for which

$F(x, \varphi, \varphi', \varphi'', \dots, \varphi^n) = 0$ For all x in I. We say that φ satisfies the differential equation on I.

For example, $y'' - 2y' + y = 0$, then $y = xe^x$ is a solution of the differential equation.

1.4.1 Classification Of A Solutions:

1) **General Solution:** If every solution of an nth-order ODE $F(x, y, y', y'', \dots, y^n) = 0$ on an interval I can be obtained from an n-parameter family $G(x, y, c_1, c_2, \dots, c_n) = 0$ by appropriate choices of the parameters $c_i, i = 1,2, \dots, n$, we then say that the family is the general solution of the DE.

2) **Particular Solution:** Any solution which is obtained from the general solution by giving particular values to the arbitrary constants.

3) **Singular Solution :** Sometimes a differential equation possesses a solution that is not a member of a family of solutions of the equation—that is, a solution that cannot be obtained by specializing any of the parameters in the family of solutions. Such an extra solution is called a singular solution.

   For example, the differential equation $\frac{dy}{dx} = xy^{\frac{1}{2}}$ possesses the one parameter family of solutions $y = \left(\frac{x^2}{4} + c\right)^2$ but $y(x) = 0$ is also a solution to this D.E. which cannot be obtained from this family of solutions by assigning the particular value to the arbitrary constant.

1.5 Formation Of A Differential Equations:

Let, $f(x, y, c_1, c_2, \dots, c_n) = 0$ be the given N- parameters family of curves. Then differential equation is obtained for this family by removing these 'n' arbitrary constant.

WORKING RULE TO FORM A DIFFERENTIAL EQUATIONS:

(1) Differentiate the given family of curves $f(x, y, c_1, c_2, \dots, c_n) = 0$ 'n' times with respect to x.

Therefore we have, $f_1(x, y, y', c_1, c_2, \dots, c_n) = 0$

$f_2(x, y, y', y'', c_1, c_2, \dots, c_n) = 0$

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$f_n(x, y, y', y'', \dots, y^{(n)}, c_1, c_2, \dots, c_n) = 0$

Now we have (n+1) equations and then with the help of these (n+1) equations we have to remove these n arbitrary constants to get the required D.E.

Examples:

(1) Find the D.E. of the family of curves $y = e^{mx}$ where m is an arbitrary constant.

Solution: $y = e^{mx}$ .....(1)

$\Rightarrow \frac{dy}{dx} = me^{mx}$ .....(2)

$\Rightarrow \frac{dy}{dx} = my$

$\Rightarrow m = \frac{1}{y}\frac{dy}{dx}$

Now substituting $m = \frac{1}{y}\frac{dy}{dx}$ in (1) we get

$y = e^{\frac{x}{y}\frac{dy}{dx}}$

$\Rightarrow \log y = \frac{x}{y}\frac{dy}{dx}$

$\Rightarrow \frac{1}{x}\log y = \frac{1}{y}\frac{dy}{dx}$

(2) Find the D.E. of all circles passing the origin and whose centers are on the x-axis.

Solution:

$x^2 + y^2 + 2gx + 2fy + c = 0$

Now, as the center on the x-axis$\Rightarrow f = 0, g = 0$

$\Rightarrow x^2 + y^2 + 2gx = 0$ ..... (1)

$\Rightarrow 2x + 2yy' + 2g = 0$

$\Rightarrow x + yy' + g = 0$

$\Rightarrow g = -x - yy'$

$\therefore$ Put $g = -x - yy'$ in (1) We get

$x^2 + y^2 + 2(-x - yy')x = 0$

$\Rightarrow x^2 + y^2 - 2x^2 - 2xyy' = 0$

Q. Form the D.E. from the relation $y = a\sin x + b\cos x + x\sin x$ where a & b are arbitrary constants.

Solution:

$y = a\sin x + b\cos x + x\sin x$ ...(1)

$\frac{dy}{dx} = a\cos x - b\sin x + x\cos x + \sin x$

$\frac{d^2y}{dx^2} = -a\sin x - b\cos x - x\sin x + \cos x + \cos x$

$= 2\cos x - (a\sin x + b\cos x + x\sin x)$

$\frac{d^2y}{dx^2} = 2cosx - y$

Which is required differential equation for the given family.

Q. Consider the differential equation, $\left|\frac{dy}{dx}\right| + |y| + a = 0$, $a \in \mathbb{R}$ then number of real valued solutions of the differential equation are as follows: