I N D E X

METRIC SPACES & TOPOLOGY

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CHAPTER 1

METRIC SPACE

A metric space is a set in which we can talk of the distance between any two of its elements. The definition below imposes certain natural conditions on the distance between the points.

1.1 DEFINITION:

Let X be a nonempty set. A function $d: X \times X \to \mathbb{R}$ is said to be a metric or a distance function on X if d satisfies the following properties:

(i) $d(x,y) \geq 0$ for all $x, y \in X$ and $d(x,y) = 0$ iff $x = y$. (non-negativity)

(ii) $d(x,y) = d(y,x)$ for all $x, y \in X$. (symmetry)

(iii) $d(x,z) \leq d(x,y) + d(y,z)$ for all $x, y, z \in X$.(triangle inequality)

The pair $\langle X, d \rangle$ is then called a metric space.

Examples of Metric Spaces:

1. $X = \mathbb{R}$ (**Set of real numbers)** and $d(x,y) = |x - y| \; \forall \; x, y \in X$
   (i) $d(x,y) = |x - y| \geq 0$
   and $d(x,y) = 0 \Leftrightarrow |x - y| = 0 \Leftrightarrow x = y$
   (iii) $d(x,y) = |x - y| = |y - x| = d(y,x)$
   (iv) $d(x,y) = |x - y| = |x - y + y - z| \leq |x - y| + |y - z|$
   $\Rightarrow d(x,y) \leq d(x,z) + d(y,z)$
   Clearly, $d(x,y)$ is metric on X and $d(x,y)$ is called usual or Euclidean metric on X.

2. $X = \mathbb{R}^2$ (**Euclidean plane**)
   Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq 2$
   And define $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2}$ is a metric on $X = \mathbb{R}^2$

3. $X = \mathbb{R}^2$
   Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq 2$
   And define $d(x,y) = |x_1 - y_1| + |x_2 - y_2|$ is a metric on $X = \mathbb{R}^2$
   This is called rectangular metric on $\mathbb{R}^2$.

4. $X = \mathbb{R}^n$ (**Euclidean space**)
   Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2, x_3, \dots \dots \dots x_n)$, $y = (y_1, y_2, y_3, \dots \dots y_n)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq n$
   And define $d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + \cdots (x_n - y_n)^2}$ is a metric on $X = \mathbb{R}^n$.

5. $X = \mathbb{R}^n$
   Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2, x_3, \dots \dots \dots x_n)$, $y = (y_1, y_2, y_3, \dots \dots y_n)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq n$
   And define $d(x,y) = |x_1 - y_1| + |x_2 - y_2| + \cdots + |x_n - y_n|$ is metric on $X = \mathbb{R}^n$.
   This is called rectangular metric on $\mathbb{R}^n$.

6. $X = \mathbb{R}^2$
   Let $x, y \in \mathbb{R}^2$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq 2$
   And define $d(x,y) = \text{Sup}\{|x_1 - y_1|, |x_2 - y_2|\}$ is a metric on $X = \mathbb{R}^2$
   This is called supremum metric or sup metric on $\mathbb{R}^2$.

7. Let $X = \mathbb{R}^n$
Let $x, y \in \mathbb{R}^n$, $x = (x_1, x_2, x_3, \dots \dots \dots x_n)$, $y = (y_1, y_2, y_3, \dots \dots y_n)$ where $x_i, y_i \in \mathbb{R}, 1 \leq i \leq n$
And define
$d(x,y) = \sup\{|x_i - y_i| : 1 \leq i \leq n\}$
$= \sup\{|x_1 - y_1|, |x_2 - y_2| \dots \dots \dots |x_n - y_n|\}$
This is called supremum metric or sup metric on $\mathbb{R}^n$.

8. $X = \mathbb{C}$ **(complex plane)**
Let $z = x + iy$ and $|z| = \sqrt{x^2 + y^2}$ (modulus of z)
Define $d(z_1, z_2) = |z_1 - z_2|$ is a metric on X.

9. $X = \mathbb{C}^n$**(complex space)**
Let $x, y \in \mathbb{C}^n$, $x = (x_1, x_2, x_3, \dots \dots \dots x_n)$, $y = (y_1, y_2, y_3, \dots \dots y_n)$ where $x_i, y_i \in \mathbb{C}, 1 \leq i \leq n$
And define $d(x,y) = \sqrt{|x_1 - y_1|^2 + |x_2 - y_2|^2 + \cdots |x_n - y_n|^2}$ is a metric on $X = \mathbb{C}^n$.

Discrete Metric:
Let X be any non-empty set and define
$d(x,y) = \begin{cases} 0, & x = y \\ \alpha, & x \neq y \end{cases}$ where α be any positive real.
(i) $d(x,y) = 0, \alpha \geq 0$ and $d(x,y) = 0 \Leftrightarrow x = y$(non-negativity)
(ii) $d(x,y) = d(y,x)$(symmetry)
$\because$ if $x = y$ then $d(x,y) = 0 = d(y,x)$ and if $x \neq y$ the $d(x,y) = \alpha = d(y,x)$
(iv) $d(x,y) \leq d(x,z) + d(y,z)$ (triangle inequality)
If $d(x,y) = 0 \Rightarrow x = y$,
$0 \leq d(x,z) + d(y,z) = \alpha + \alpha = 2\alpha$
If $d(x,y) = \alpha \Rightarrow x \neq y$,
$\alpha \leq d(x,z) + d(y,z) = 0 + \alpha = \alpha$ if $x = z$
$\alpha \leq d(x,z) + d(y,z) = \alpha + \alpha = 2\alpha$ if $x \neq y \neq z$
Clearly, $d(x,y)$ is a metric and $d(x,y)$ is called discrete metric and (X,d) is called discrete metric space.

Note:
In general, we write discrete metric on X as

$$d(x,y) = \begin{cases} 0, & x = y \\ 1, & x \neq y \end{cases}$$

which is special case for $\alpha = 1$ and we can define discrete metric on any non-empty set like $\mathbb{R}, \mathbb{Q}, \mathbb{N}, \mathbb{Z}$.

***Q*.** Which of the following is/are metric on $\mathbb{R}$
(a) $d(x,y) = \min(x,y)$
(b) $d(x,y) = |x - 2y| + |2y - x|$
(c) $d(x,y) = |x^2 - y^2|$
(d) $d(x,y) = |x^3 - y^3|$
Solution:
(a) if $x = -1, y = 1$
$d(x,y) = \min(-1, 1)$
$= -1 < 0$ and we know that $d(x,y) \geq 0$
Then (a) is not metric on $\mathbb{R}$.
(b) $x = 1, y = \frac{1}{2}$
Then $d(x,y) = 0$ but $1 \neq \frac{1}{2}$ and $d(x,y) = 0$ iff $x = y$.
Then (b) is not metric on $\mathbb{R}$.

(c) $x = 1, y = -1$
$d(x,y) = |x^2 - y^2| = 0$ but $1 \neq -1$ and $d(x,y) = 0$ iff $x = y$.
Then (c) is not metric on $\mathbb{R}$.
(d) $d(x,y) = |x^3 - y^3|$
(i) $d(x,y) \geq 0$
(ii) $d(x,y) = |x^3 - y^3| = |y^3 - x^3| = d(y,x)$
(iii) $d(x,y) = |x^3 - y^3| = |x^3 - z^3 + z^3 - y^3| \leq |x^3 - z^3| + |z^3 - y^3| = d(x,z) + d(z,y)$
Clearly, $d(x,y) \leq d(x,z) + d(z,y)$
(iv) $d(x,y) = |x^3 - y^3| = 0$
$\Leftrightarrow x^3 - y^3 = 0$
$\Leftrightarrow (x-y)(x^2 + xy + y^2) = 0$
$\Leftrightarrow (x-y) = 0$ or $(x^2 + xy + y^2) = 0$
If $(x-y) = 0$
$\Leftrightarrow x = y$
If $(x^2 + xy + y^2) = 0$
then $x = \frac{-y \pm \sqrt{y^2 - 4y^2}}{2}$
$x = \frac{y(-1 \pm i\sqrt{3})}{2}$ but here x, y lies in complex number. So $(x^2 + xy + y^2) = 0$
iff $x = 0, y = 0$
then also $x = y$.
Hence, $d(x,y) = |x^3 - y^3|$ is metric on X.

***Q*.** For $\alpha \in \mathbb{R}$. Let $[\alpha]$ denote greatest integer smaller than or equal to α
Define $d: \mathbb{R} \times \mathbb{R} \to [0, \infty)$ by $d(x,y) = [|x-y|], x, y \in \mathbb{R}$ then which of the following are true?
(a) $d(x,y) = 0 \Leftrightarrow x = y,\ x, y \in \mathbb{R}$
(b) $d(x,y) = d(y,x), x, y \in \mathbb{R}$
(c) $d(x,y) \leq d(x,z) + d(z,y), x, y, z \in \mathbb{R}$
(d) d is not a metric on $\mathbb{R}$
Solution: Let $x = 0.9$ and $y = 0.4$
Then $[|x-y|] = [0.9 - 0.4] = [0.5] = 0$
$d(0.9, 0.4) = [|x-y|] = 0$
But $0.9 \neq 0.4$
Clearly, (a) is false⇒ (d) is true
(b) $d(x,y) = [|x-y|] = [|y-x|]$
$= d(y,x)$
⇒ (b) is true.
(c) Now let $x = 1.3, y = 0.2$
And $z = 0.4$
$d(x,y) = [|x-y|] = [|1.3 - 0.2|] = [1.1] = 1$
$d(x,z) = [|x-z|] = [|1.3 - 0.4|] = [0.9] = 0$
$d(z,y) = [|z-y|] = [|0.4 - 0.2|] = [0.2] = 0$
$\Rightarrow d(1.3, 0.2) \neq d(1.3, 0.4) + d(0.4, 0.2)$
Therefore (c) is false.

Function Space $C[a,b]$

1. Let $X = C[a,b]$ is defined as set of all continuous maps from $[a,b]$ to $\mathbb{R}$ $i.e$
$C[a,b] = \{f: [a,b] \longrightarrow \mathbb{R} : f \text{ is continuous}\}$.
Define $d(f,g) = \sup_{x \in [a,b]} \{|f(x) - g(x)|: f, g \in C[a,b]\}$
This is called **supremum metric** on $C[a,b]$

(i) $d(f,g) = \sup_{x\in[a,b]} \{|f(x) - g(x)|\} \geq 0$

And $d(f,g) = 0$

$\Leftrightarrow \sup_{x\in[a,b]} \{|f(x) - g(x)|\} = 0$

$\Leftrightarrow |f(x) - g(x)| = 0 \; \forall \; x \in [a,\ b]$

$\Leftrightarrow f(x) = g(x) \; \forall \; x \in [a,b]$

$\Leftrightarrow f = g$

(ii) $d(f,g) = \sup_{x\in[a,b]} \{|f(x) - g(x)|\}$

$= \sup_{x\in[a,b]} \{|g(x) - f(x)|\}$

$= d(g,f)$

(iii) $d(f,g) = \sup_{x\in[a,b]} \{|f(x) - g(x)|\}$

$\Rightarrow |f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)|$

$\Rightarrow |f(x) - g(x)| \leq \sup_{x\in[a,b]} |f(x) - h(x)| + \sup_{x\in[a,b]} |h(x) - g(x)|$

$\Rightarrow \sup_{x\in[a,b]} |f(x) - g(x)| \leq \sup_{x\in[a,b]} |f(x) - h(x)| + \sup_{x\in[a,b]} |h(x) - g(x)|$

$\Rightarrow \; d(f,g) \leq d(f,h) + d(h,g)$

Clearly, this is a metric on $C[a,b]$.

2. Let $X = C[a,b]$ is defined as set of all continuous maps from $[a,b]$ to $\mathbb{R}$

Define $(f,g) = \int_a^b |f(x) - g(x)|\, dx$.

(i) $d(f,g) = \int_a^b |f(x) - g(x)| dx \geq 0$

and $d(f,g) = 0 \Leftrightarrow \int_a^b |f(x) - g(x)| dx = 0$

$\Leftrightarrow |f(x) - g(x)| = 0 \; \forall \; x \in [a,b]$

$\Leftrightarrow f(x) = g(x) \; \forall \; x \in [a,b]$

$\Leftrightarrow f = g$

(ii) $d(f,g) = \int_a^b |f(x) - g(x)|\, dx$

$= \int_a^b |g(x) - f(x)| = d(g,f)$

(iii) $d(f,g) = \int_a^b |f(x) - g(x)| dx$

$\Longrightarrow |f(x) - g(x)| \leq |f(x) - h(x)| + |h(x) - g(x)| \;\; \forall \; f,g,h \in C[a,b]$

$$\Longrightarrow \int_a^b |f(x) - g(x)|\, dx \leq \int_a^b |f(x) - h(x)| dx \; + \int_a^b |h(x) - g(x)| dx$$

$\Longrightarrow d(f,g) \leq d(f,h) + d(h,g).$

Clearly, this defines a metric on X.

Sequence Space $\boldsymbol{l^\infty}$

Let $X = \boldsymbol{l^\infty}$,Set of all bounded sequences of real numbers; that is, every element of X is a real sequence

$$x = (x_1, x_2, \ldots\ldots) \quad \text{briefly} \quad x = (x_i)$$

such that for all $i = 1, 2, \ldots$ we have

$$|x_i| \leq r_x$$

where r_x is a real number which may depend on x, but does not depend on i.

Define a metric on X

$$d(x,y) = \sup_{i\in\mathbb{N}} |x_i - y_i|$$

where $y = (y_i) \in X$ and $\mathbb{N} = \{1, 2, \ldots.\}$, and sup denotes the supremum (least upper bound).

The metric space thus obtained is generally denoted by l^∞.

l^∞ is a sequence space because each element of X (each point of X) is a sequence.

Space l^p

Let $p \geq 1$ be a fixed real number.

Let $X = l^p$,set of all real sequences $x = (x_i) = (x_1, x_2, \ldots.)$ such that $|x_1|^p + |x_2|^p + \cdots.$ converges; thus

$$\sum_{i=1}^{\infty} |x_i|^p < \infty \qquad (p \geq 1, fixed)$$

and the metric is defined by

$$d(x,y) = \left(\sum_{i=1}^{\infty} |x_i - y_i|^p \right)^{1/p}$$

where $y = (y_i)\ and\ \sum |y_i|^p < \infty$.

In the case $p = 2$ we have the famous Hilbert sequence space l^2 with metric defined by

$$d(x,y) = \sqrt{\sum_{i=1}^{\infty} |x_i - y_i|^2}$$

This space was introduced and studied by D. Hilbert (1912) in connection with integral equations and is the earliest example of what is now called a Hilbert space. You will study more about this space in functional analysis.

Q. Let (X,d) be a metric space and define $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$ then $d'(x,y)$ is metric on X. [**T**/F]

Solution:

(i) $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} \geq 0$ as $d(x,y) \geq 0$ and $1 + d(x,y) \geq 1$

and $d'(x,y) = 0 \Leftrightarrow \frac{d(x,y)}{1+d(x,y)} = 0$

$\Leftrightarrow d(x,y) = 0$

$\Leftrightarrow x = y$

(ii) $d'(x,y) = \frac{d(x,y)}{1+d(x,y)} = \frac{d(y,x)}{1+d(y,x)} = d'(y,x)$

(iii) $d(x,y) \leq d(x,z) + d(z,y)$

[if $a, b \geq\ a \leq b$

$\Rightarrow a + ab \leq b + ab$

$\Rightarrow a + ab \leq b + ab$

$\Rightarrow a(1+b) \leq b(1+a)$

$\Rightarrow \frac{a}{1+a} \leq \frac{b}{1+b}$]

$\Rightarrow \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)+d(z,y)}{1+d(x,z)+d(z,y)}$

$= \frac{d(x,z)}{1+d(x,z)+d(z,y)} + \frac{d(z,y)}{1+d(x,z)+d(z,y)}$

$\leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$

$\Rightarrow \frac{d(x,y)}{1+d(x,y)} \leq \frac{d(x,z)}{1+d(x,z)} + \frac{d(z,y)}{1+d(z,y)}$

$\Rightarrow d'(x,y) \leq d'(x,z) + d'(z,y)$

Q. Which of the following is/are metric on $\mathbb{R}$?

(a) $d(x,y) = \frac{||x|-|y||}{1+|x||y|}$

(b) $d(x,y) = \sqrt{|x-y|}$

(c) $d(x,y) = |f(x) - f(y)|$, Where f is strictly monotonically increasing function $f: \mathbb{R} \to \mathbb{R}$.

(d) None of these

Solution:

(a) $d(x,y)=\frac{||x|-|y||}{1+|x||y|}$

For $x=1, y=-1$

Then $d(x,y)=0$ but $x\neq y$ so option (a) is False.

(b) $d(x,y)=\sqrt{|x-y|}$

(i) $d(x,y)\geq 0$

(ii) $d(x,y)=0 \Leftrightarrow \sqrt{|x-y|}=0 \Leftrightarrow x=y$

(iii) $d(x,y)=\sqrt{|x-y|}=\sqrt{|y-x|}=d(y,x)$

(iv) $d(x,y)\leq d(x,z)+d(z,y)$

$\Rightarrow \left(\sqrt{|x-z|}+\sqrt{|z-y|}\right)^2=|x-z|+|z-y|+2\sqrt{|x-z|}\sqrt{|z-y|}$

$\geq |x-y|+2\sqrt{|x-y|}\sqrt{|y-z|}\geq |x-y|$

$\Rightarrow \sqrt{|x-z|}+\sqrt{|z-y|}\geq \sqrt{|x-y|}$

$\Longrightarrow d(x,z)+d(z,y)\geq d(x,y)$

so, option (b) is True.

(c) $d(x,y)=|f(x)-f(y)|$, Where f is strictly monotonically increasing function $f:\mathbb{R}\to\mathbb{R}$ so, f is one-one function.

(i) $d(x,y)\geq 0$

(ii) $d(x,y)=|f(x)-f(y)| \Leftrightarrow 0$

$\Leftrightarrow f(x)=f(y)$

$\Leftrightarrow x=y$

(iii) $d(x,y)=|f(x)-f(y)|=|f(y)-f(x)|=d(y,x)$

(iv) we know that $|f(x)-f(y)|\leq |f(x)-f(z)|+|f(z)-f(x)|$ implies

$d(x,y)\leq d(x,z)+d(z,y)$ so, option (c) is true.

Result:

1. Let $\langle X,d\rangle$ be any metric space and if $d'(x,y)=K.d_1(x,y)$, where K is positive real number then $d'(x,y)$ is also a metric on X.
2. Let $\langle X,d_1\rangle$ and $\langle X,d_2\rangle$ be any two metric spaces then $d_3(x,y)=d_1(x,y)+d_2(x,y)$ be also a metric on X.

Q. Let d_1, d_2 be the following metrices on $\mathbb{R}^n$

$$d_1(x,y)=\sum_{i=1}^{n}|x_i-y_i|$$

$$d_2(x,y)=\left(\sum_{i=1}^{n}|x_i-y_i|^2\right)^{1/2}$$

Then decide which of the following is a metric on $\mathbb{R}^n$

(a) $d(x,y)=\frac{d_1(x,y)+d_2(x,y)}{1+d_1(x,y)+d_2(x,y)}$

(b) $d(x,y)=d_1(x,y)-d_2(x,y)$

(c) $d(x,y)=d_1(x,y)+d_2(x,y)$

(d) $d(x.y)=e^{\pi}\,d_1(x,y)+e^{-\pi}\,d_2(x,y)$

Solution:

If d_1 and d_2 are metrics then d_1+d_2 is a metric, $\alpha\, d_1$ is a metric for $\alpha>0$ and $\frac{d_1}{1+d_1}$ is a metric.

This proves (a), (c) and (d) are true.

(b) is not true because, for example,

If $x=(0,0,\ldots,0)$ and $y=(1,0,\ldots,0)$

then $d_1(x,y)=1=d_2(x,y)$

$\Rightarrow d_1(x,y) - d_2(x,y) = 0$ but $x \neq y$

Alternatively:

On $\mathbb{R}^2$ given that

$$d_1(x,y) = \sum_{i=1}^{n} |x_i - y_i|$$

And $d_2(x,y) = (\sum_{i=1}^{n} |x_i - y_i|^2)^{\frac{1}{2}}$ are metrics

If d_1 and d_2 are metrics on X then

1. $ad_1 + bd_2$ is also metric on X for $a > 0, b > 0$ and
2. if d is metric on X then $\frac{d}{1+d}$ is also metric on X

We the help of these two results we say that (a), (c), (d) are true.

Q. Which of the following are metrics on $C[0,1] = \{f: [0,1] \to \mathbb{R} \ \ is\ a\ continuous\ function\}$.

(a) $d(f,g) = \sup(|f(x) - g(x)|: x \in [0,1])$

(b) $d(f,g) = \inf(|f(x) - g(x)|: x \in [0,1])$

(c) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$

(d) $d(f,g) = \sup(|f(x) - g(x)|: x \in [0,1]) + \int_0^1 |f(x) - g(x)|\, dx$

Solution:

(a) $d(f,g) = \sup(|f(x) - g(x)|: x \in [0,1])$ is a metric on $C[0,1]$.

(b) Take $f(x) = x$ for $x \in [0,1]$ and $g(x) = 2x$ for $g \in [0,1]$

Now, $d(f,g) = \inf\{|f(x) - g(x)|: x \in [0,1]\}$

$= \inf\{|x - 2x|: x \in [0,1]\}$

$= \inf\{|x|: x \in [0,1]\} = 0$

But $f(x) \neq g(x)$

So, $d(f,g) = \inf(|f(x) - g(x)|: x \in [0,1])$ is not metric.

(c) $d(f,g) = \int_0^1 |f(x) - g(x)|\, dx$ is metric on $C[0,1]$

(d) If (d_1, X) and (d_2, X) are metric spaces then $(d_1 + d_2, X)$ is again metric space

So, $(f,g) = \sup(|f(x) - g(x)|: x \in [0,1])$

$$+ \int_0^1 |f(x) - g(x)|\, dx$$

Clearly, this is a metric on $C[0,1]$.

Q. Let d and d' be metrics on a non-empty set X . Then which of the following are metrics on X?

(a) $p_1(x,y) = d(x,y) + d'(x,y)$ for all $x, y \in X$

(b) $p_2(x,y) = d(x,y)d'(x,y)$ for all $x, y \in X$

(c) $p_3(x,y) = \max\{d(x,y), d'(x,y)\}$ for all $x, y \in X$

(d) $p_4(x,y) = \min\{d(x,y), d'(x,y)\}$ for all $x, y \in X$

Solution:

option (a)

If d_1 and d_2 are metrics on X then $k_1 d_1 + k_2 d_2$ is also a metric on X if both k_1 and k_2 are non-negative and at least one of k_1 and k_2 is positive.

So, $p_1(x,y) = d(x,y) + d'(x,y)$ is metric on X.

option (b)

Product of two metric need not be metric

Take $X = \mathbb{R}$

then $d(x,y) = |x - y|$

and $d'(x,y) = \min\{1, |x - y|\}$

The triangle inequality should hold for $p_2(x,y) = d(x,y)d'(x,y)$ for all points $x, y, z \in \mathbb{R}$ for it to be metric
Consider the points

$$x = 1, y = 2 \text{ and } z = 2.6$$

We have $p_2(1,2.6) = d(1,2.6)d'(1,2.6) = |1-2.6|.\min\{1,|1-2.6|\} = 1.6$

$$p_2(1,2) + p_2(2,2.6) = d(1,2)d'(1,2) + d(2,2.6)d'(2,2.6)$$
$$= |1-2|.\min\{1,|1-2|\} + |2-2.6|.\min\{1,|2-2.6|\} = 1 + 0.6 \times 0.6 = 1.36$$

and $1.36 < 1.6$ thus triangle inequality does not hold.
Option (c)
$p_3(x,y) = \max\{d(x,y), d'(x,y)\};\ \forall (x,y) \in X$
Clearly,
(i) $p_3(x,y) \geq 0;\ \forall x, y \in X$
(ii) $p_3(x,y) = 0$
$\Leftrightarrow d(x,y) = 0 \text{ and } d'(x,y) = 0$
$\Leftrightarrow x = y$
(iii) Clearly, $p_3(x,y) = \max\{d(x,y), d'(x,y)\};\ \forall x, y \in X$
$= \max\{d(y,x), d'(y,x)\};\ \forall x, y \in X = p_3(y,x)$
(iv) Given $x, y, z \in X$
$p_3(x,z)$ is either equal to $d(x,z)$ or $d'(x,z)$
Suppose first that

$$p_3(x,z) = d(x,z)$$

By triangle inequality of d, we have

$$p_3(x,z) = d(x,z) \leq d(x,y) + d(y,z)$$
$$\leq p_3(x,y) + p_3(y,z)$$

Similarly, the triangle inequality holds if

$$p_3(x.y) = d'(x,y)$$

Hence, p_3 is metric on X.
Option (d)
$p_4(x,y) = \min\{d(x,y), d'(x,y)\}$ for all $x, y \in X$
Take $X = \mathbb{R}^2$

$$d(x,y) = \sqrt{(x_1-y_1)^2 + \frac{1}{100}(x_2-y_2)^2}$$

$$d'(x,y) = \sqrt{\frac{1}{100}(x_1-y_1)^2 + (x_2-y_2)^2}$$

So, let $x = (1,0),\ y = (0,1),\ z = (0,0)$

$$d(x,y) = \sqrt{(1-0)^2 + \frac{1}{100}(0-1)^2}$$
$$= \sqrt{1 + \frac{1}{100}}$$

and,

$$d'(x,y) = \sqrt{\frac{1}{100}(1-0)^2 + (0-1)^2}$$
$$= \sqrt{\frac{1}{100} + 1}$$

$$p_4(x,y) = \min\{d(x,y), d'(x,y)\} = \sqrt{1 + \frac{1}{100}} > 1$$