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GROUP THEORY

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SECTION-I

GROUP THEORY

CHAPTER 1

SETS, RELATIONS & FUNCTIONS

1.1 Definition Set:

A set is a well-defined collection of objects.

(Here well-defined means there is no confusion between inclusion & exclusion of the objects)

Some important sets:

$\mathbb{N}$ = The set of natural nos.

$\mathbb{W}$ =The set of whole nos.

$\mathbb{Z}$ =The set of integers

$\mathbb{Z}^+$= The set of +ve integers

$\mathbb{Z}^-$=The set of negative integers

$\mathbb{Q}$ =The set of rational numbers

$\mathbb{Q}^C$=The set of irrational numbers

$\mathbb{R}$ =The set of real number

$\mathbb{C}$ =The set of complex numbers

1.2 Representation of a Sets:

1.2.1 Roaster form/ Tabular form: In this method a set is represented by listing all its elements.

Ex: - If A be the set of natural number less than 10 then we can write A = {1,2,3,4,5,6,7,8,9}.

- In Roaster form elements can't be repeated.

Ex: If A be the set of all letters in the word CHANDAN, then we can write in Roaster form as

$A = \{C, H, A, N, D\}$

- In Roaster form order of arrangement doesn't matter

Ex: {1,2,3} represents the same set as {1,3,2}

1.2.2 Set-Builder Form: In this form elements of the set are not listed but these are represented by some common property.

Ex: If A be the set of natural number less than 10 then we can write A in set builder form as

$A = \{x: x \in \mathbb{N} \,\&\, 1 \leq x < 10\}$.

1.3 Classification of Sets:

1.3.1 Definition Finite Set: A set is called finite if its elements can be counted.

Ex: A = $\{x : x\ is\ a\ student\ of\ IFAS\}$

1.3.2 Definition Infinite Set: A set is called Infinite if its elements can't be counted.

Ex: $\mathbb{N}$

1.3.3 Definition **Empty set / Null set**: A set which has no elements is said to be an empty/null/void set and denoted by ϕ.

Ex: $A = \{x: x \in \mathbb{R} \,\&\, x^2 + 1 = 0\}$

1.3.4 Definition **Singleton set**: A set which has only one element is known as singleton set.

Ex: A = $\{x : x\ is\ even\ prime\}$

1.3.5 Definition **Equal set**: If the Sets A and B have same elements then they are called equal.

Ex: A = {1,2,3} and B = {1,2,3} then A = B

1.3.6 Definition **Equivalent set**: If the Sets A and B have same no. of elements then they are called equivalent set.

Ex: A = {1,2,3} and B = $\{a, b, c\}$ then A is equivalent to B or A ≈ B

- Equal set always equivalent but equivalent set need not be equal
- Two sets A and B are called equivalent sets if there exist one to one correspondence between them.

1.3.7 Definition Disjoint Sets: Two sets are said to be disjoint if they don't have any common element.

Ex: - A = $\{x: x\ is\ solution\ of\ x^2 + 6x + 8 = 0\}$ & B = {10, 11} Then A and B are called disjoint set.

1.4 Definition Subset:

A set A is called subset of a set $B\ if\ \forall\ x \in A \implies x \in B$ & written as A ⊆ B.

Some Facts About Subset:

- Every set is the subset of itself
- If A ⊆ B and B ⊆ A then A = B
- Empty set is the subset of every set

1.4.1 Definition Proper Subset: If $A \subseteq B\, s.t. A \neq \phi \text{ and } A \neq B$, Then A is said to be proper subset of B & B is called superset of A.

$i.e.$ we can write A ⊂ B or B ⊃ A

Ex: $A = \{[e^x] | \, x \in \mathbb{R}\}, [.]$ denotes the greatest integer function. $i.e.$ A ⊂ $\mathbb{R}$

- A & ϕ are called **improper** subsets of A & all the other subsets are called proper subset

1.5 Power Set:

A set of all subsets of set A is called power set of A & denoted by P(A).

⟶ If $|A| = n$, then the number of subsets of $A = |P(A)| = 2^n$

1.6 Universal Set:

A universal set is a set which contains all the elements or objects of other concerned sets, including its own elements. denoted by the symbol U.

Ex: if X = {2, 4, 6, 8}, Y= {3, 7, 9, 11} & Z = {4, 8, 11} Then we can define the universal set of X, Y, and Z as

U = {2, 3, 4, 6, 7, 8, 9, 11}.

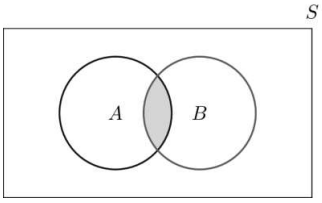
1.7 Intersection of Two Set:

The intersection of two sets A and B, denoted by $A \cap B$, & defined as $A \cap B == \{x | x \in A \text{ and } x \in B\}$

$i.e.$ intersection consists of all elements that are both in A and B.

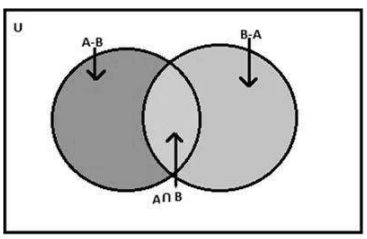
Ex: If $A = \{1,2\}$ & $B = \{2,3\}$ then $A \cap B = \{1,2\} \cap \{2,3\} = \{2\}$.

In the given figure the intersection of sets A and B is shown by the shaded area in the Venn diagram.



1.8 Difference of Two Set:

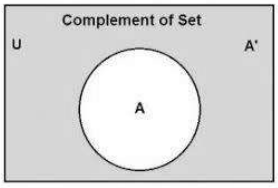
If A and B are two sets then the difference between two sets denoted by A \ B and defined as $A \setminus B = \{x | \, x \in A \text{ but } x \notin B\}$



1.9 Complement of a Set:

If U is a universal set and A be any subset of U then the set of all members of the universal set U which are not the elements of A is called complement of A. It is denoted by A′ & defined as A′ = {x : x ∈ U and x ∉ A}

Also, it can be said that the difference of the universal set U and the subset A gives us the complement of set A.



1.10 Union of Two Set:

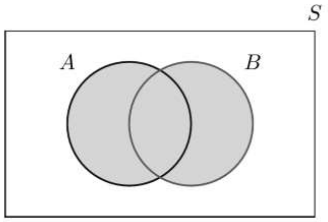
The union of two sets is a set containing all elements that are in A or in B (possibly both) denoted by A ∪ B & defined as $A \cup B = \{x | x \in A \text{ or } x \in B\}$

Ex: $A = \{1,2\}$ & $B = \{2,3\}$ then $A \cup B = \{1,2\} \cup \{2,3\} = \{1,2,3\}$.

$i.e.$ we can write x ∈ (A∪B) if and only if (x ∈ A) or (x ∈ B).

- A∪B = B∪A

In the given figure the union of sets A and B is shown by the shaded area in the Venn diagram.



Similarly,

We can define the union of three or more sets.

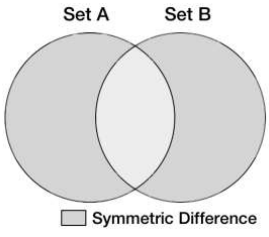
In particular, if $A_1, A_2, A_3 \ldots . A_n$ are n sets, then we can write this union more

compactly by $U_{i=1}^{n} A_i$ & also for $A_1, A_2, A_3 \ldots . A_n$...their union can be written as $U\, A_i \text{ or } U_{i=1}^{\infty} A_i$

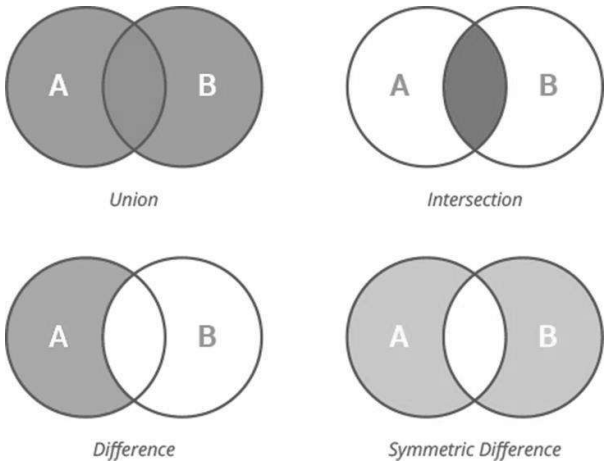
1.11 Symmetric Difference of Two Set:

Symmetric difference between two sets A & B is denoted by

A ∆ B & defined by $A \triangle B = (A \cup B) \setminus (A \cap B)$



Venn diagram for Union, Intersection, Difference and Symmetric difference:



1.12 Cartesian Product of Two Sets: If A and B are any two sets then Cartesian product of two sets denoted by A × B & defined as

$A \times B = \{(a, b) : a \in A \& b \in B\}$

Ex: $A = \{1,2,3\}$ & $B = \{a, b\}$ then

$A \times B = \{(1, a), (1, b), (2, a), (2, b), (3, a), (3, b)\}$

B × A = $\{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Important observation:

- If $|A| = n$ & $|B| = m$ then $|A \times B| = mn$
- If $A \neq \phi$ & $B \neq \phi$ then $A \times B \neq B \times A$ iff $A \neq B$
- If $A \neq \phi$ & $B \neq \phi$ then $A \times B = B \times A$ iff $A = B$
- If either $A = \phi$ or $B = \phi$ then $A \times B = B \times A = \phi$
- If $|A| = n$ & $|B| = m$ then $|P(A \times B)| = 2^{mn}$

1.13 Relation: If A and B are any two sets then the subset of A × B represents a relation (say R) defined from A to B.

Ex: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$

- If $|A| = n$ & $|B| = m$ then total no. of distinct relation defined from Set A to Set B = Total no. of subsets of $A \times B = |P(A \times B)|$

1.13.1 Domain of relation: The set of all first elements of the ordered pairs in a relation R from a set A to a set B is called the domain of the relation R.

Ex: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$

1.13.2 Codomain of relation: If R is the relation defined from set A to set B then the whole set B is called the codomain of the relation R

Ex: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by R = {(x, y) : y = x + 1 }

1.13.3 Range of the relation: The set of all second elements in a relation R defined from a set A to a set B is called the range of the relation R.

- Range ⊆ Codomain

Ex: Let A = {1, 2, 3, 4, 5, 6}. Define a relation R from A to A by $R = \{(x, y) : y = x + 1\}$

1.14 Types of Relation

1.14.1 Empty Relation: $\phi = S \subseteq A \times B$ is called an empty Relation defined from Set A to Set B.

1.14.2 Identify Relation: if A be any non-empty set the Relation I_A is called $\forall\, a \in A, (a, a) \in I_A$ & $\forall\, a \neq b, (a, b) \notin I_A$.

Ex: $A = \langle 1, 2, 3 \rangle$

$I_A \subseteq A \times A$ & $I_A = \langle (1, 1), (2, 2), (3, 3) \rangle$

- If $|A| = n, |I_A| = n$

1.14.3 Reflexive Relation: If A be any set & if S_A is a Relation defined on A s.t. $I_A \subseteq S_A$ then S is called a reflexive Relation defined on A.

Ex. $A = \langle 1, 2, 3 \rangle$

$I_A = \langle (1, 1), (2, 2), (3, 3) \rangle$

$S_A = \langle (1, 1), (2, 2), (3, 3), (1, 2) \rangle$

- Identity Relation ⇒ Reflexive relation
- If $|A| = n, |S_A| \geq |I_A| = n$
- Total number of Reflexive relations defined on set A, if $|A| = n$ is equal to $2^{n^2 - n}$

1.14.4 Irreflexive Relation: If A be any set, $|A| = n$ & S_A is a relation defined on A is called Irreflexive if $S_A \cap I_A = \phi$

Ex: $A = \langle 1, 2, 3 \rangle$

$S'_A = \langle (1, 2), (1, 3) \rangle$

If $A = \langle 1, 2, 3 \rangle$

$R \subseteq A \times A$ $s.t.$ $R = \langle (1, 1), (1, 2), (1, 3) \rangle$

- Total number of Irreflexive Relation defined on A = $2^{n^2 - n}$
- ∃ some Relations which are neither Reflexive nor Irreflexive.

1.14.5 Symmetric Relation: If A be any set & S be a Relation defined on set A $s.t.$ $\forall\, (a, b) \in S, (b, a) \in S$ then S is called symmetric Relation defined on A

Ex: $A = \langle 1, 2, 3 \rangle$

$S \subseteq A \times A, S = \langle (1,1),(1,3),(3,1) \rangle$

If $|A| = n$ then Total no. of symmetric Relation defined on set $A = 2^{\frac{n(n+1)}{2}} = 2^{\Sigma n}$

1.14.6 Transitive Relation: If A be a set and S is a Relation defined on A $s.t. \forall (a,b) \in S$ & $(b,c) \in S$ $then$ $(a,c) \in S$.

1.15 Equivalence Relation:

If A be a set and S is a relation defined on A $s.t. S$ is Reflexive symmetric and Transitive.

Ex: Let $a, b \in \mathbb{N}$, aRb if $a+b$ is even

Check whether R is an equivalence relation or not?

Solution:

(i) Reflexive:- let $a \in \mathbb{N}$ show aRa
$a + a = 2a \Rightarrow 2a$ is even
aRa
$\Rightarrow R$ is reflexive

(ii) Symmetric:- If aRb then bRa
$\because aRb \Rightarrow a+b$ is even
$\Rightarrow b+a$ is even
$\Rightarrow bRa$

(iii) Transitive:- Let aRb & bRc then aRc
$\because aRb \Rightarrow a+b$ is even
$\Rightarrow a+b = 2m$
$\because bRc \Rightarrow b+c$ is even
$\Rightarrow b+c = 2n$

Consider $a+b+b+c = a+2b+c$
$\Rightarrow a+2b+c = 2m+2n$
$\Rightarrow a+c = 2m-2n-2b$
$a+c = 2(m+n-b)$
$a+c = 2k$ where $k = m+n-b$
$\Rightarrow a+c$ is even

Above relation is an equivalence relation

Ex: Let $a, b \in \mathbb{N} - \{1\}$, aRb if $\gcd(a,b) \neq 1$

R is not an equivalence relation, since it is not transitive

Counter example: $(3,6),(6,4)$

Ex: S is a Relation defined on set $A \subseteq \mathbb{R}$ $s.t$ $(a,b) \in S$ iff $a - b \in A \subseteq \mathbb{R}$ then S is an equivalence relation if $A = \mathbb{Z}$

1.15.1 Equivalence Class: Let S be an Equivalence Relation on a set A and $a \in A$. Then the set defined and denoted as $cl(a) = \{x \in A | (x,a) \in S\}$ and is called an equivalence class of $a \in A$ by the relation S.

- The Equivalence classes are either disjoint or identical.

1.15.2 Quotient Set: Let A be any non-empty set and S be an Equivalence relation defined on A. Then the set of all Equivalence Classes corresponding to the relation S is denoted by $\frac{A}{S}$

Equivalence class:- Let S be an equivalence relation on A and $a \in A$. Then the equivalence class of a denoted by $cl(a)$ or $[a]$ and defined as

$cl(a) = [a] = \{x \in A; xRa\}$

Ex: Let $a, b \in \mathbb{N}$, aRb if $a+b$ is even

$\Rightarrow [a] = cl(a) = [a] = \{x \in \mathbb{N}; xRa\}$

$cl(1) = [1] = \{x \in \mathbb{N}: xR1\}$

$= \{x \in \mathbb{N}: x+1$ is even$\}$

$= \{x \in \mathbb{N}: x$ is odd$\}$

$= \{1,3,5,7,\dots\}$

$cl(2) = [2] = \{a \in \mathbb{N}: xR2\}$

$= \{x \in \mathbb{N}: x+2$ is even$\}$

$= \{x \in \mathbb{N}: x$ is even$\}$

$= \{2,4,6,8,\dots\}$

$\frac{A}{S} = \{cl(1), cl(2)\}$

1.15.3 Partition: Let S be a set then a partition of S is a set of subsets $\mathbb{S}$ of S such that

- $\mathbb{S}$ is pairwise disjoint: $\forall S_1, S_2 \in \mathbb{S}: S_1 \cap S_2 = \phi$ when $S_1 \neq S_2$
- The union of $T \in \mathbb{S}$ forms the whole set S $i.e.$ $\cup T = S$
- None of the elements of $\mathbb{S}$ is empty: $\forall T \in \mathbb{S}$ $i.e. T \neq \emptyset$

or

A partition of S is a set of non-empty subsets $\mathbb{S}$ of S such that each element of S lies in exactly one element of $\mathbb{S}$.

- The elements $S_1, S_2, \dots \in \mathbb{S}$ are known as the **components** of the partition.
- No. of equivalence relations on $A =$ Total number of partitions of A
- $P(r)$ represents the number of equivalence relations or no. of partitions on a set with r elements.

Then considering $P(0) = 1$ we can find the partition with $P(n+1) = \sum_{r=0}^{n} {}^{n}C_r P(r)$

i.e.
$P(1) = \mathbf{1}$
$P(2) = 1 \rightarrow \mathbf{2}$
$P(3) = 2 \rightarrow 3 \rightarrow \mathbf{5}$
$P(4) = 5 \rightarrow 7 \rightarrow 10 \rightarrow \mathbf{15}$
$P(5) = 15 \rightarrow 20 \rightarrow 27 \rightarrow 37 \rightarrow \mathbf{52}$