I N D E X

LINEAR ALGEBRA

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CHAPTER 1

MATRICES & THEIR PROPERTIES

1.1 BASIC DEFINITIONS

1.1.1 Matrix: A rectangular arrangement of mn elements in m-rows and n-columns is called a matrix.

- If a matrix contain m-rows and n-columns then it's order is $m \times n$.
- Matrices are usually denoted by capital letters $A, B, C, \ldots$ and elements or number or entries in matrices are denoted by a_{ij} or b_{ij}.
- In short we write

$$A = [a_{ij}]_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}.$$

- A matrix whose elements is come from the set F is known as F-matrix.

  Ex: If $A = [a_{ij}]_{m\times n}$ is real matrix, then $a_{ij} \in \mathbb{R}\ \forall$ i, j. Then A is called a real matrix or $\mathbb{R}$- matrix.

Q. How many matrices of order 3×3 are there over $\mathbb{Z}_7$? where $\mathbb{Z}_7 = \{0, 1, 2, 3, 4, 5, 6\}$.

(a) 9^7 (b) 7^9

(c) 7^7 (d) 9^9

Solution: Total number of elements in matrix $= 9$. Since each entry has 7 choices, therefore,

Required number of matrices

$= 7 \times 7 \times \ldots 7(9 - times) = 7^9$

1.2 TYPES OF MATRICES

1.2.1 Square Matrix: A matrix in which number of rows is same as the number of columns, is called square matrix.

Ex: $A = \begin{bmatrix} 1 & 2 \\ 3 & 1 \end{bmatrix}_{2\times 2}$ is a square matrix of order 2.

1.2.2 Principal Diagonal Line of a Square Matrix: The line along which elements $a_{ii}, i = 1, 2, \ldots, n$ lies is known as principal diagonal line of $A = [a_{ij}]_{n\times n}$.

Ex: Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}_{n\times n}$. Then elements along the principal diagonal line of A are $a_{11}, a_{22}, \ldots, a_{nn}$.

1.2.3 Diagonal Elements of a Square Matrix: If A is a square matrix of order $n \times n$ $\left(i.e.\ A = [a_{ij}]_{n\times n}\right)$ then number of elements along principal diagonal line of A is n and the elements along principal diagonal line are called as diagonal elements.

1.2.4 Super Diagonal Line of a Square Matrix: The line along the elements $a_{ij}, j = i + 1\ \forall\ i = 1, .., n - 1,$ is known as super diagonal line of $A = [a_{ij}]_{n\times n}$.

Note: If $A = [a_{ij}]_{n\times n}$ then the number of elements along super diagonal line are $n - 1$.

1.2.5 Row Matrix or Row Vector: A matrix having 1 row but any number of columns is known as row matrix.

Ex: $A = [a_{11}\ a_{12} \ldots a_{1n}]_{1\times n}$ is a row vector or a row matrix of order $1 \times n$.

1.2.6 Column Matrix or Column Vector: A matrix having 1 column but any number of rows is known as column matrix or a column vector.

Ex: $A = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{bmatrix}_{n\times 1}$ is a column vector or a column matrix of order $n \times 1$.

1.2.7 Zero Matrix or Null Matrix: A matrix whose all elements are zero is called as zero matrix or null matrix.

Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a 3×2 zero or null matrix.

1.2.8 Diagonal matrix: $A = [a_{ij}]_{n\times n}$ is said to be a diagonal matrix if $a_{ij} = 0\ \forall\ i \neq j$. If $d_1, d_2, \ldots d_n$ are the diagonal entries of some diagonal matrix A of order n then A can be denoted as $= diag[d_1, d_2, \ldots d_n]$.

Ex: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$ is a diagonal matrix of order 3 with diagonal entries 1,2 & 3.

1.2.9 Properties: If $A = diag[a_1, a_2, \dots, a_n]$ and $B = diag[b_1, b_2, \dots, b_n]$ are two diagonal matrices of order n and k is any scalar(by scalar we mean k is an element of the respective field) then.

- $kA = diag[ka_1, ka_2, \dots, ka_n]$
- $A + B = diag[a_1 + b_1, a_2 + b_2, \dots, a_n + b_n]$
- $AB = diag[a_1b_1, a_2b_2, \dots, a_nb_n]$
- $A^m = diag[a_1^m, a_2^m, \dots, a_n^m]$

1.2.10 Identity Matrix: An identity matrix is a diagonal matrix whose each diagonal entry is 1. Identity matrix of order n is denoted by I_n. If there is no concern about order, we denote identity matrix by I.

Ex: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}_{3\times3}$ is an identity matrix of order 3.

1.2.11 Scalar Matrix: A diagonal matrix with same diagonal entries is called a scalar matrix. Diagonal entries of scalar matrices can be anything but each entry has to be same. Identity matrix is a particular type of scalar matrix with each diagonal entry 1. If A is a scalar matrix of order n then

$$A = \begin{bmatrix} k & 0 & 0 & 0 \\ 0 & k & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & k \end{bmatrix} = k\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix} = kI_n$$

1.2.12 Triangular Matrix

(A) Upper Triangular Matrix: Let $A = [a_{ij}]_{n\times n}$ then A is said to be upper triangular if $a_{ij} = 0 \; \forall \; i > j$.

(B) Lower Triangular Matrix: Let $A = [a_{ij}]_{n\times n}$ then A is said to be upper triangular if $a_{ij} = 0 \; \forall \; i < j$.

Ex: $U = \begin{bmatrix} 1 & -1 & 3 \\ 0 & 2 & 4 \\ 0 & 0 & 5 \end{bmatrix}$ is a upper triangular matrix and $L = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ -2 & 3 & 4 \end{bmatrix}$ is a lower triangular matrix.

Note: Diagonal matrices are both UTM(upper triangular matrix) and LTM(lower triangular matrix).

1.2.13 Properties: Let A and B are UTM (or LTM) of order n and k is any scalar then

- kA is UTM (or LTM)
- $A + B$ is also UTM $(or\ LTM)$
- AB is also UTM $(or\ LTM)$
- A^m is also UTM $(or\ LTM)$
-

1.2.14 Sub Matrix: A matrix obtained by eliminating some rows or columns or both rows and columns from a given matrix A is called a sub matrix of A.

Ex: $B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ is a submatrix obtained by eliminating 2nd row and 3^{rd} column of $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 6 \\ 3 & 4 & 5 \end{bmatrix}$.

1.2.15 Equality Of Matrices:- Two matrices A and B are said to be equal if

- they are of same order,
- their corresponding elements are same

i.e., $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{m\times n}$ then $A = B \;\; iff \;\; a_{ij} = b_{ij}$ $\forall\; i, j$.

Ex: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}_{2\times2}$, $B = \begin{bmatrix} s & t \\ u & v \end{bmatrix}_{2\times2}$ then $A = B \; iff \;\; a = s, b = t, c = u \;\&\; d = v$.

1.3 ALGEBRA OF MATRICES

1.3.1 Addition of Matrices: Two matrices $A = [a_{ij}]_{m\times n}$ and $B = [b_{ij}]_{m\times n}$ are comformable for addition if they are of same order and their sum is given by

$$A + B = [a_{ij} + b_{ij}]_{m\times n}$$

Ex: If $A = \begin{bmatrix} 1 & 2 \\ 5 & 4 \end{bmatrix}, B = \begin{bmatrix} -1 & 8 \\ 5 & -4 \end{bmatrix}$ then

$$A + B = \begin{bmatrix} 1-1 & 2+8 \\ 5+5 & 4-4 \end{bmatrix} = \begin{bmatrix} 0 & 10 \\ 10 & 0 \end{bmatrix}$$

1.3.2 Properties:

- Matrix addition is commutative *i.e.,* $A + B = B + A$ for any matrix A and B of same order.
- Matrix addition is associative *i.e.,* $(A + B) + C = A + (B + C)$ for any three matrices $A, B \;\&\; C$ of same order .

1.3.3 Multiplication of a Matrix by a Number or Scalar: If $A = [a_{ij}]_{m\times n}$ and k is any number then $kA = [ka_{ij}]_{m\times n}$.

Ex: If $A = \begin{bmatrix} 1 & -1 & 4 \\ 2 & 3 & -2 \end{bmatrix}, k = -3$ then

$$kA = (-2)A = \begin{bmatrix} -3\times1 & -3\times-1 & -3\times4 \\ -3\times2 & -3\times3 & -3\times-2 \end{bmatrix} = \begin{bmatrix} -3 & 3 & -12 \\ -6 & -9 & 6 \end{bmatrix}.$$

Note:

- $k(A + B) = kA + kB$
- $(k_1 + k_2)A = k_1A + k_2A$
- $(k_1\, k_2)A = k_1(k_2A)$

1.3.4 Multiplication of Matrices: A matrix A is conformable for premultiplication to a matrix B if number of columns in A is equal to number of rows in B. If $A = [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ then the product $AB = C = [c_{ij}]_{m\times p}$, where $c_{ij} = \sum_{k=1}^{n} a_{ik}b_{kj}$.

Ex: If $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}_{2\times3}$ and $B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix}_{3\times2}$ then

$$AB = \begin{bmatrix} a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31} & a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32} \\ a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31} & a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32} \end{bmatrix}_{2\times 2}$$

$$= \begin{bmatrix} \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} b_{11} + \begin{bmatrix} a_{12} \\ a_{21} \end{bmatrix} b_{21} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} b_{31} & \begin{bmatrix} a_{11} \\ a_{21} \end{bmatrix} b_{12} + \begin{bmatrix} a_{12} \\ a_{21} \end{bmatrix} b_{22} + \begin{bmatrix} a_{13} \\ a_{23} \end{bmatrix} b_{32} \end{bmatrix}$$

$$= \begin{bmatrix} A \begin{bmatrix} b_{11} \\ b_{21} \\ b_{31} \end{bmatrix} & A \begin{bmatrix} b_{12} \\ b_{22} \\ b_{332} \end{bmatrix} \end{bmatrix}$$

1.3.5 Linear Combination of Rows and Columns of a Matrix: If $R_1, R_2, \dots, R_m$ and $C_1, C_2, \dots, C_n$ are rows and columns of $m \times n$ matrix A respectively. Then linear combination of rows of A is defined by $\alpha_1 R_1 + \alpha_2 R_2 + \cdots + \alpha_m R_m$ where $\alpha_1, \alpha_2, \dots, \alpha_m$ are some scalars. Similarly, for some scalars $\beta_1, \beta_2, \dots, \beta_n$, linear combination of columns of A is defined by $\beta_1 C_1 + \beta_2 C_2 + \cdots + \beta_n C_n$.

We will discuss about linear combination of vectors in detail in chapter 2 vector space.

Remark: In the above example observe that, rows of AB are the linear combination of rows of B, and columns of AB are the linear combination of columns of A.

1.3.6 Properties:

- In general two matrices need not to commute with each other *i.e.* in general $AB \neq BA$.
- Multiplication of matrices is always associative *i.e.,* $A(BC) = (AB)C$ where A, B, C are any three matrices which are conformable for multiplication.
- If A and B are two matrices such that $AB = 0$, it does not necessarily imply that either $A = 0$ or $B = 0$ *i.e.,* we can have two non-zero matrices A and B such that $AB = 0$.

  Ex:- Let $A = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 3 \end{bmatrix}$. Note that both A and B are non-zero but $AB = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is a zero matrix.
- Let $A = [a_{ij}]_{m\times n}$ and $b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$ then i^{th} entry of column vector Ab is the sum of elements of i^{th} row of A.

1.3.7 Proposition: Let A and B are two matrices such that s_1 and s_2 are the sum of each row of A and B respectively then $s_1 s_2$ will be the sum of each row of matrix AB.

Let $= [a_{ij}]_{m\times n}$, $B = [b_{ij}]_{n\times p}$ and $b = \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix}_{n\times 1}$ then

$$ABb = A(Bb) = A \begin{bmatrix} s_2 \\ s_2 \\ \vdots \\ s_2 \end{bmatrix} \{from\ property\ (iv)\ above\}$$

$$= s_2 A \begin{bmatrix} 1 \\ 1 \\ \vdots \\ 1 \end{bmatrix} = s_2 \begin{bmatrix} s_1 \\ s_1 \\ \vdots \\ s_1 \end{bmatrix} = \begin{bmatrix} s_1 s_2 \\ s_1 s_2 \\ \vdots \\ s_1 s_2 \end{bmatrix}$$

So $s_1 s_2$ is the sum of each row of AB.

1.3.8 Corollary 1: Let A and B are tow matrices such that t_1 and t_2 are the sum of each column of A and B respectively then $t_1 t_2$ will be the sum of each column of matrix AB.

Note: Columns of A are rows of A^T, now use above proposition for A^T.

1.3.9 Corollary 2: Let $A = [a_{ij}]_{n\times n}$ such that each row (column) sum is a then for each $n \in \mathbb{N}$ sum of each row (column) of A^n is a^n.

1.4 SOME IMPORTANT TERMS

1.4.1 Invertible matrix: A square matrix A of order n is said to be invertible if there exist another matrix B of order n such that $AB = BA = I_n$, where I_n is the identity matrix of order n. If such matrix B exist we say that A is the inverse of B and B is the inverse of A or A and B are inverse of each other.

Ex: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$. Note that $AB = BA = I_2$, therefore A and B are inverse of each other.

Q. Let $A = [a_{ij}]_{n\times n}$ and $x = [b_{ij}]_{n\times 1}$ such that $b_{ij} = 1\ \forall\ i, j$ then $x^T Ax$ represents ?

(a) $\sum_{i=1}^n \sum_{j=1}^n a_{ij}$ (b) $\sum_{i=1}^n a_{ij}$

(c) $\sum_{j=1}^n a_{ij}$ (d) $\sum_{i=1}^n \sum_{j=1}^n a_{ij}^2$

Solution: By above proposition $Ax = \begin{bmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{bmatrix}$.

$$\Rightarrow x^T Ax = [1 \quad 1 \quad \cdots \quad 1] \begin{bmatrix} \sum_{j=1}^n a_{1j} \\ \sum_{j=1}^n a_{2j} \\ \vdots \\ \sum_{j=1}^n a_{nj} \end{bmatrix} = \sum_{i=1}^n \sum_{j=1}^n a_{ij} =$$

Sum of each entry of A.

Q. Let A, B *be* $n \times n$ matrices such that $BA + B^2 = I - BA^2$ where I is the $n \times n$ identity matrix. Which of the followings always true?

(a) A is non- singular (b) B is non-singular

(c) $A + B$ is non-singular (d) AB is non-singular

Solution: (b)

$BA + B^2 = I - BA^2$

$\Rightarrow B(A + B + A^2) = I$

$\Rightarrow |B(A + B + A^2)| = |I|$

$\Rightarrow |B||(A + B + A^2)| = 1$

$\Rightarrow \det B \neq 0$ and B is invertible.

Thus option (b) is correct.

1.4.2 Trace of Matrix: If $A = [a_{ij}]_{n\times n}$ then sum of all diagonal elements in A is called trace of A and it is denoted by $trace(A)$ or $tr(A)$.

i.e. $tr(A) = a_{11} + a_{22} + \cdots + a_{nn}$

Ex: Let $A = \begin{bmatrix} 11 & 12 & 13 \\ 9 & 14 & 15 \\ 0 & 1 & -10 \end{bmatrix}_{3\times 3}$ then

$tr(A) = 11 + 14 + (-10)$

$= 15$

1.4.3 Properties: Let $A = [a_{ij}]_{n\times n}$, $B = [b_{ij}]_{n\times n}$ and k is any scalar then

- $tr(kA) = k\, tr(A)$
- $tr(A + B) = tr(A) + tr(B)$
- $tr(AB) = tr(BA)$

1.4.4 Transpose of matrix: Let $A = [a_{ij}]_{m\times n}$ then transpose of A is denoted by A^T *or* A' and is defined as $A^T = [a_{ji}]_{n\times m}$.

Ex: Let $A = \begin{bmatrix} 2 & -1 \\ i & -i \\ 5 & 1 \end{bmatrix}_{3\times 2}$ then $A^T = \begin{bmatrix} 2 & i & 5 \\ -1 & -i & 1 \end{bmatrix}_{2\times 5}$.

1.4.5 Properties: Let A and B are two matrices and k is any number then

- $(A^T)^T = A$
- $(kA)^T = kA^T$
- $(A + B)^T = A^T + B^T$
- $(AB)^T = B^T A^T$
- $(A_1 A_2 \ldots . A_n)^T = A_n^T A_{n-1}^T \ldots . A_2^T\, A_1^T$
- For a square matrix A, $(A^n)^T = (A^T)^n$.

1.4.6 Proposition: Let $A = [a_{ij}]_{m\times n}$ then trace of AA^T equal to sum of square of each element of A *i.e.,* $trace(A^T A) = \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2$

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$ then

$A^T = \begin{bmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & \vdots & \vdots \\ a_{1n} & a_{2n} & \ldots & a_{mn} \end{bmatrix}_{n\times m}$

Now

$trace(AA^T)$

$= \sum_{j=1}^{n} a_{1j}\, a_{1j} + \sum_{j=1}^{n} a_{2j}\, a_{2j} + \cdots + \sum_{j=1}^{n} a_{mj}\, a_{mj}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\, a_{ij}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n}(a_{ij})^2$.

1.4.7 Conjugate of Matrix: A matrix obtain from a given matrix $A = [a_{ij}]_{m\times n}$ by replacing each element of A by it's complex conjugate is known as conjugate of matrix and it is denoted by $\bar{A}$.

Ex: Let $A = \begin{bmatrix} 1+i & 1-i \\ 2-i & 3+i \end{bmatrix}$ then $\bar{A} = \begin{bmatrix} 1-i & 1+i \\ 2+i & 3-i \end{bmatrix}$

1.4.8 Properties: Let A and B are matrices and k is any scalar then

- $\overline{(\bar{A})} = A$
- $\overline{(kA)} = \bar{k}\,\bar{A}$
- $\overline{(A+B)} = \bar{A} + \bar{B}$
- $\overline{AB} = \bar{A}\bar{B}$
- If A is a square matrix and $n \in N$ then $(\bar{A})^n = \overline{(A^n)}$

1.4.9 Conjugate Transpose of a Matrix: If $A = [a_{ij}]_{m\times n}$ then transpose conjugate of A is denoted by A^θ or A^H or A^* and defined as $A^\theta = \overline{(A^T)} = (\bar{A})^T = [\overline{a_{ji}}]_{n\times m}$.

Ex: Let $A = \begin{bmatrix} 1-i & 1+i \\ 2-i & 3+i \end{bmatrix}$ then $A^\theta = \begin{bmatrix} 1+i & 2+i \\ 1-i & 3-i \end{bmatrix}$.

1.4.10 Proposition: Let $A = [a_{ij}]_{m\times n}$ then trace of AA^θ equal to sum of square of modulus of each element of A *i.e.,* $trace(A^\theta A) = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2$.

Let $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}_{m\times n}$ then

$A^\theta = \begin{bmatrix} \overline{a_{11}} & \overline{a_{21}} & \cdots & \overline{a_{m1}} \\ \overline{a_{12}} & \overline{a_{22}} & \cdots & \overline{a_{m2}} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a_{1n}} & \overline{a_{2n}} & \ldots & \overline{a_{mn}} \end{bmatrix}_{n\times m}$

Now

$trace(AA^\theta) = \sum_{j=1}^{n} a_{1j}\,\overline{a_{1j}} + \sum_{j=1}^{n} a_{2j}\,\overline{a_{2j}} + \cdots + \sum_{j=1}^{n} a_{mj}\,\overline{a_{mj}}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n} a_{ij}\,\overline{a_{ij}}$

$= \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2 \ \left\{since\ \ a_{ij}\overline{a_{ij}} = |a_{ij}|^2\right\}$

Q. If $A = \begin{bmatrix} 1 & i \\ -i & -1 \end{bmatrix}$ then $trace(AA^\theta)$ =?

(a) 1 (b) i

(c) -1 (d) 4

Solution: We have $trace(AA^\theta) = \sum_{i=1}^{m}\sum_{j=1}^{n}|a_{ij}|^2$

$= |1|^2 + |i|^2 + |-i|^2 + |-1|^2$

$= 1 + 1 + 1 + 1$

$= 4$

1.5 SOME SPECIAL MATRICES

1.5.1 Symmetric Matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is said to be symmetric if $a_{ij} = a_{ji}\ \forall\ i, j$ *i.e., A* is symmetric if $A^T = A$.

Ex: I_n, identity matrix of order n is a symmetric matrix.

1.5.2 Properties:

- If A and B are two symmetric matrices and they follow the commutative property, i.e. $AB = BA$, then the product of AB is symmetric.
$$(AB)^T = B^T A^T = BA = AB$$
- If matrix A is symmetric then A^n is also symmetric, where n is an integer.
$$(A^n)^T = (A^T)^n = (A)^n$$
- If A and B are symmetric matrices of order n and k is any number then $(A + B)$ and kA are also symmetric.
$$(A+B)^T = A^T + B^T = A + B,$$
$$(kA)^T = kA^T = kA$$
- If A is a symmetric matrix of order n and P is any square matrix of order n then $P^T AP$ is also symmetric.
$$(P^T AP)^T = P^T A^T (P^T)^T = P^T AP$$
- If A is any square matrix $A + A^T$ is always symmetric matrix.
$$(A + A^T)^T = A^T + (A^T)^T = A^T + A$$
- If A is any matrix then AA^T and $A^T A$ is always symmetric.
$$(A^T A)^T = A^T (A^T)^T = A^T A$$
Similarly we can check for AA^T.

1.5.3 Skew Symmetric Matrix: A square matric $A = [a_{ij}]_{n\times n}$ is said to be skew symmetric if $a_{ij} = -a_{ji}\ \forall\ i, j$ *i.e.,* A is skew symmetric if $A^T = -A$.

Observe that the only matrix which is Symmetric and skew-symmetric both is the zero matrix.

Ex: $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is a skew symmetric matrix of order 2 as $A^T = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -A$.

Properties:

- Diagonal entries of skew-symmetric matrices are 0.
$$a_{ii} = -a_{ii}\ \forall\ i \Rightarrow 2a_{ii} = 0 \Rightarrow a_{ii} = 0\ \forall\ i$$
- If A is skew-symmetric and k is any scalar then kA is also skew symmetric.
$$(kA)^T = kA^T = k(-A) = -kA$$
- If A and B are skew-symmetric then $(A + B)$ is also skew-symmetric.
$$(A+B)^T = A^T + B^T = -A - B = -(A+B)$$
- If A and B are skew symmetric with $AB = BA$ then AB is symmetric.
$$(AB)^T = B^T A^T = (-B)(-A) = BA = AB$$
- If A is a skew-symmetric matrix then A^m is skew symmetric matrix if m is odd and symmetric matrix if m is even.
$$(A^m)^T = (A^T)^m = (-A)^m = \begin{cases} A^m, & m\ is\ even \\ -A^m, & m\ is\ odd \end{cases}$$
- If A is a skew symmetric matrix and P is any square matrix of order n then $P^T AP$ is also skew-symmetric matrix.
$$(P^T AP)^T = P^T A^T (P^T)^T = P^T(-A)P = -(P^T AP)$$

1.5.4 Proposition: Every square matrix can be uniquely expressed as the sum of symmetric and skew symmetric matrix.

If we write $A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$, note that $\frac{1}{2}(A + A^T)$ is a symmetric matrix and $\frac{1}{2}(A - A^T)$ is skew symmetric matrix.

Ex: Let $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ then $\frac{1}{2}(A + A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\frac{1}{2}(A - A^T) = \frac{1}{2}\left\{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\right\} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$.Now $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$, note that $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ is symmetric and $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ is skew symmetric.

1.5.5 Hermitian matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is said to be Hermitian (or self-adjacent) if $a_{ij} = \overline{a_{ji}}\ \forall\ i, j$ or $A^\theta = A$. Note that for diagonal entries of Hermitian matrix, $a_{ii} = \overline{a_{ii}}$, that is diagonal entries of Hermitian matrices are real. Every real symmetric matrix is Hermitian.

Ex: $A = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix}$ is a Hermitian matrix, as $A^\theta = \overline{(A^T)} = \begin{bmatrix} 1 & 1+i \\ 1-i & 2 \end{bmatrix} = A$.

1.5.6 Properties:

- Let A is a Hermitian matrix and k is any scalar then kA is Hermitian if k is real and skew-Hermitian if k is purely imaginary.
$$(kA)^\theta = \bar{k}A^\theta = \bar{k}A$$
$$= \begin{cases} kA, & if\ k\ is\ real \\ -kA, & if\ k\ is\ purely\ imaginary \end{cases}$$
- If A and B are Hermitian matrices then $A + B$ is also Hermitian.
$$(A+B)^\theta = A^\theta + B^\theta = A + B\ \{\because A^\theta = A, B^\theta = B\}$$
- If A and B are Hermitian matrices such that $AB = BA$ then AB is Hermitian.
$$(AB)^\theta = B^\theta A^\theta = BA = AB$$
- If A is Hermitian then $A^m\ \forall\ m \in \mathbb{N}$ is also Hermitian.
$$(A^m)^\theta = (A^\theta)^m = A^m$$
- For any square matrix P, If A is Hermitian then $P^\theta AP$ is also Hermitian.
$$(P^\theta AP)^\theta = P^\theta A^\theta (P^\theta)^\theta = P^\theta AP$$
- For any square matrix A, $A^\theta A$ and AA^θ are always Hermitian.
$$(A^\theta A)^\theta = A^\theta (A^\theta)^\theta = A^\theta A$$
Similarly we can check for AA^θ.
- For any square matrix A, $A + A^\theta$ is always Hermitian.
$$(A + A^\theta)^\theta = A^\theta + (A^\theta)^\theta = A^\theta + A = A + A^\theta$$

1.5.7 Skew-Hermitian matrix: A square matrix $A = [a_{ij}]_{n\times n}$ is said to be skew-Hermitian if $a_{ij} = -\overline{a_{ji}}\ \forall\ i,j$ or $A^\theta = -A$.

Note that for diagonal entries of skew-Hermitian matrix, $a_{ii} = -\overline{a_{ii}} \Rightarrow a_{ii} + \overline{a_{ii}} = 0 \Rightarrow Re(a_{ii}) = 0$, that is diagonal entries of skew-Hermitian matrices are either purely imaginary or zero. Every real skew-symmetric matrix is skew-Hermitian. The only matrix which is Hermitian and skew-Hermitian both is the only zero matrix.

Ex: Let $A = \begin{bmatrix} 0 & 1-i \\ -1-i & i \end{bmatrix}$. Note that is a skew-Hermitian, as $A^\theta = \overline{(A^T)} = \begin{bmatrix} 0 & -1+i \\ 1+i & -i \end{bmatrix} = -A$.

1.5.8 Properties:

- Let A is a Skew-Hermitian matrix and k is any scalar then kA is skew-Hermitian if k is real and Hermitian if k is purely imaginary.
  $(kA)^\theta = \bar{k}A^\theta = \bar{k}(-A) =$
  $\begin{cases} -kA, & if\ k\ is\ real \\ kA\ , if\ k\ is\ purely\ imaginary \end{cases}$
- If A and B are skew-Hermitian matrices then $A+B$ is also skew-Hermitian.
  $(A+B)^\theta = A^\theta + B^\theta = -A - B\ = -(A+B)$
  $\{\because A^\theta = -A,\ B^\theta = -B\}$
- If A and B are skew-Hermitian matrices such that $AB = BA$ then AB is Hermitian.
  $(AB)^\theta = B^\theta A^\theta = (-B)(-A) = BA = AB$
- If A is skew-Hermitian then $A^m\ \forall\ m \in \mathbb{N}$ is Hermitian if n is even and skew-Hermitian if n is odd.
  $(A^m)^\theta = \left(A^\theta\right)^m = (-A)^m = \begin{cases} A^m, if\ m\ is\ even \\ -A^m, if\ m\ is\ odd \end{cases}$
- For any square matrix P, If A is skew-Hermitian then $P^\theta AP$ is also skew-Hermitian.
  $\left(P^\theta AP\right)^\theta = P^\theta A^\theta \left(P^\theta\right)^\theta = P^\theta(-A)P = -P^\theta AP$
- For any square matrix , $A - A^\theta$ is always skew-Hermitian matrix.
- Every skew-Hermitian matrix A can be written as $A = iB$ where B is some Hermitian matrix.
  A is skew-Hermitian so $A^\theta = -A$.
  $\Rightarrow A^\theta = i^2A = i(iA) = iB\ \{B = iA\}$
  Find that B is Hermitian (Use A is skew-Hermitian).

1.5.9 Proposition: Every square matrix can be uniquely expressed as the sum of Hermitian and skew- Hermitian matrix.

We can write $A = \frac{1}{2}\left(A + A^\theta\right) + \frac{1}{2}\left(A - A^\theta\right)$, note that $\frac{1}{2}\left(A + A^\theta\right)$ is Hermitian and $\frac{1}{2}\left(A - A^\theta\right)$ is skew-Hermitian.

1.5.10 Orthogonal Matrix:- A square matrix A of order n is said to be orthogonal if $A^T = A^{-1}\ i.e., AA^T = I_n$, where I_n is the identity matrix of order n. If $c_1, c_2 \ldots c_n$ are the columns of orthogonal matrix A then

$c_i^T \cdot c_j = \begin{cases} 0, if\ i \neq j \\ 1, if\ i = j \end{cases}$.

Ex: Let $A = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}$. Note that A is an orthogonal matrix, as $AA^T = A^TA = I_2$.

1.5.11 Properties:

- If A and B are orthogonal matrices of order n then AB and BA both are orthogonal matrices.
  $(AB)(AB)^T = (AB)(B^TA^T) = A(BB^T)A^T = AI_nA^T = AA^T = I_n\ \ \{\because AA^T = I_n\ , BB^T = I_n\}$,
  $similarly\ \ (AB)^TAB = I_n$.
  Check for BA by yourself.
- If A is orthogonal matrix of order n then A^m is also orthogonal from any $m \in \mathbb{Z}$.
  $(A^m)(A^m)^T = (A^m)(A^T)^m = (A^{m-1}A)(A^T(A^T)^{m-1}) = (A^{m-1})(AA^T)(A^T)^{m-1} = (A^{m-1})I_n(A^T)^{m-1} = (A^{m-1})(A^T)^{m-1}$, keep doing this, at the end we will get identity and then repeat the same process for $(A^m)^T(A^m)$.

1.5.12 Unitary matrix : A Square matrix A of order n is said to be unitary if $A^{-1} = A^\theta\ i.e., AA^\theta = A^\theta A = I_n$. If $c_1, c_2 \ldots c_n$ are the columns of orthogonal matrix A then $c_i^\theta \cdot c_j = \begin{cases} 0, if\ i \neq j \\ 1, if\ i = j \end{cases}$

Ex: Let $U = \begin{bmatrix} \frac{1}{2} - \frac{i}{2} & \frac{1}{2} + \frac{i}{2} \\ \frac{1}{2} + \frac{i}{2} & \frac{1}{2} - \frac{i}{2} \end{bmatrix}$. Find that U is an unitary matrix, as $U^TU = UU^T = I_2$.

1.5.13 Properties:

- If A and B are unitary matrices of order n then AB and BA both are unitary matrices.
- If A is unitary matrix of order n then A^m is also unitary from any $m \in \mathbb{Z}$.
  We can prove both the properties in same way as we did for orthogonal matrix.

1.5.14 Idempotent Matrix: A square matrix A is said to be idempotent is it satisfies $A^2 = A$. Idempotent matrix is also known as projection matrix.

Ex: Let $A = \begin{bmatrix} 3 & -6 \\ 1 & -2 \end{bmatrix}$. Find that A is an idempotent matrix, as it satisfies $A^2 = A$.

Q. If A is an invertible idempotent matrix then
(a) A is identity matrix
(b) A is a skew-symmetric matrix
(c) A is a skew-Hermitian matrix
(d) None of these

Solution: Since A is idempotent so $A^2 = A$, also A is invertible so A^{-1} exists.
Now multiply A^{-1} both side we have $A = I$. Hence option (a) is correct.

Q. Number of diagonal, idempotent matrices of order n are?
(a) 2^n (b) n^2
(c) 3^n (d) 1

Solution: Let A is a diagonal matrix such that $A = diag[a_1, a_2, \dots a_n]$ and $A^2 = A$.
$\Rightarrow A^2 = diag[a_1^2, a_2^2, \dots\dots a_n^2]$
Now $A^2 = A \Leftrightarrow diag[a_1.a_2, \dots . a_n] = diag[a_1^2, a_2^2, \dots\dots a_n^2]$
$\Leftrightarrow a_i = a_i^2$
$\Leftrightarrow a_i^2 - a_i = 0$
$\Leftrightarrow a_i(a_i - 1) = 0$
$\Leftrightarrow a_i = 0, 1$
Therefore for each diagonal entry we have two choices. So number of diagonal idempotent matrices is $2 \times 2 \times \dots \times 2(n - times) = 2^n$. Hence option (a) is correct.

1.5.15 Properties:

- If A is idempotent matrix then A^T & A^θ are also idempotent.
  $(A^T)^2 = (A^2)^T = I^T = I$, similar argument will work for A^θ.
- If A is idempotent matrix and I is identity matrix then $I - A$ is idempotent matrix.
  $(I - A)^2 = I^2 + A^2 - 2IA = I + A - 2A = I - A$ $\{\because A^2 = A\}$
- If A and B are idempotent then $A + B$ is idempotent $AB = BA = 0$.
  $(A + B)^2 = A^2 + B^2 + AB + BA = A + B$
  $\Leftrightarrow AB = BA = 0$ $\{\because A^2 = A,\ B^2 = B\}$
- If A and B are idempotent matrices such that $AB = BA$ then AB is also idempotent.
  $(AB)^2 = A^2B^2 = AB\{\because A^2 = A, B^2 = B\}$
- If A is idempotent then A^n is also idempotent $\forall\ n \in \mathbb{N}$.
- $(A^n)^2 = (A^2)^n = A^n$ $\{\because A^2 = A\}$

1.5.16 Involutory Matrix: A square matrix A is said to be Involutary if it satisfies $A^2 = I$.

Ex: Let $A = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$. Note that A is involutory matrix as it satisfies $A^2 = I$.

1.5.17 Nilpotent matrix: A square matrix A is said to be a nilpotent if there exist a natural number k such that $A^k = \mathbf{0}$. The smallest k such that $A^k = \mathbf{0}$ is know as Index of nilpotent matrix A. That is, if Index of A is k then $A^m \neq \mathbf{0}$ for all $m < k$.

Ex: Let $A = \begin{bmatrix} 5 & -3 & 2 \\ 15 & -9 & 6 \\ 10 & -6 & 4 \end{bmatrix}$. Check that $A^2 = \mathbf{0}$, where 0 is 3×3 zero matrix, so A is nilpotent matrix with index 2.

1.5.18 Properties:

- If A is a nilpotent matrix then A^T & A^θ are also nilpotent matrix.
  A is nilpotent means there exist $k \in \mathbb{Z}$ such that $A^k = \mathbf{0}$. Now $(A^T)^k = (A^k)^T = \mathbf{0}$. So A^k is also nilpotent and same argument will work for A^θ.
- If A is nilpotent then cA is also nilpotent for any scalar c.
  $$(cA)^k = c^k A^k = \mathbf{0}\ if\ A^k = \mathbf{0}$$
- If A and B are nilpotent matrices such that $AB = BA$ then $A + B$ is also nilpotent matrix.
  Suppose that index of A and B are k_1 and k_2 respectively with $k_1 \geq 1$ and $k_2 \geq 1$ then $A^{k_1} = B^{k_2} = 0$. As $AB = BA$, we can expand $(A + B)^{k_1 + k_2}$ with the Binomial theorem:
  $$(A + B)^{k_1 + k_2} = \sum_{i=0}^{k_1 + k_2} \binom{k_1 + k_2}{i} A^i B^{k_1 + k_2 - i}$$

Note: that if $i \geq k_1$, then $A^i = A^{k_1} A^{i - k_1} = 0$ and if $i \leq k_1$, then $k_1 + k_2 - i \geq k_2$, and so $B^{k_1 + k_2 - i} = 0$. Therefore $(A + B)^{k_1 + k_2} = 0$. Hence $A + B$ is nilpotent.

- $(k_1 + k_2)$ may not be the index of $A + B$, if k_1 is the index of A and k_2 is the index of B.
  Take $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -1 \\ 0 & 0 \end{bmatrix}$ then $A + B = A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Observe that A and both are nilpotent with the same index 2 while $A + B$ is nilpotent with index 1.
- $Index(A + B) \leq Index(A) + Index\ (B) - 1$.
  Hint- Use property third above.
- If A and B are nilpotent matrices such that $AB = BA$ then AB is also nilpotent matrix with $Index(AB) \leq \min\{Index\ (A), index\ (B)\}$.
  Hint- Let index of A is k_1 and index of B is k_2 with $k_1 < k_2$ then $(AB)^{k_1} = (A^{k_1})(B^{k_1}) = \mathbf{0}$.
- If A is nilpotent matrix with index k, then A^m is also nilpotent of with index $\left\lceil \frac{k}{m} \right\rceil, m \in \mathbb{Z}$, where $\lceil . \rceil$denotes smallest integer function.

Q. If A is nilpotent matrix of index 101 then index of A^7 is equal to
(a) 101 (b) 7
(c) 14 (d) 15

Solution: $Index(A^7) = \left\lceil \frac{m}{k} \right\rceil = \left\lceil \frac{101}{7} \right\rceil = [14.4 \dots] = 15.$
Hence option (d) is correct.

Q. Let A and B are two nilpotent matrices which commute with each other with $Index\ (A) = k_1,\ Index(B) = k_2$ such that $k_1 - k_2 > 0$ then which of the following is/are correct?

(a) $Index(AB) \leq k_1$

(b) $Index(AB) \leq k_2$

(c) $Index(AB) \leq k_1 k_2$

(d) None of these

Solution: Since $k_1 - k_2 > 0$ so $k_1 > k_2$ so $\min\{k_1, k_2\} = k_2$. $Index(AB) \leq \min\{Index(k_1), Index(k_2)\} = \min\{k_1, k_2\} = k_2$ i.e., $Index(AB) \leq k_2$.

Hence option$(a), (b)\&(c)$ are correct.

1.5.19 Normal Matrices: A square matrix A is said to be normal if $AA^\theta = A^\theta A$.

Ex:

- Unitary matrices are normal as $AA^\theta = I = A^\theta A$.
- Hermitian matrices are normal as $AA^\theta = A^2 = A^\theta A\ \{\because A^\theta = A\}$.
- Skew-Hermitian matrices are normal as $AA^\theta = -A^2 = A^\theta A\ \{\because A^\theta = -A\}$.

1.6 DETERMINANT

1.6.1 Definition: The determinant is defined as a scalar value which is associated with the square matrix. If A is a square matrix, then the determinant of A is represented by $|A|$ or $\det A$. Determinant of 2×2 matrix $\begin{bmatrix} a_1 & b_1 \\ a_2 & b_2 \end{bmatrix}$ is defined as $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = a_1 b_2 - a_2 b_1$ and determinant of 3×3 matrix $\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ is defined as $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1)$.

Ex: Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ then

$$|A| = \begin{vmatrix} 1 & 2 \\ 3 & 4 \end{vmatrix} = 4 \times 1 - 2 \times 3 = 4 - 6 = -2,$$

$$|B| = \begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{vmatrix} = 1(2 \times 1 - 3 \times 1) - 2(2 \times 2 - 3 \times 1) + 3(2 \times 3 - 3 \times 1) = 1 \times (-1) - 2 \times 1 + 3 \times 3 = 6.$$

Note: Let $A = [a_{ij}]_{n \times n}$ then the determinant of A is given as $|A| = \sum_{i=1}^{n} (-1)^{1+i} a_{1i} |A_{1i}|$.

Where A_{1i} is the submatrix obtained by omitting 1^{st} row and i^{th} column of A. Above formula of determinant is expansion corresponding to the first row of A, we can also expand corresponding to any row or column of a accordingly.

1.6.2 Minor: A minor of a matrix A is the determinant of a square submatrix of A obtained by deleting some set of rows and some set of columns.

Ex: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $k = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ is a minor of A obtained by deleting first row and third column.

1.6.3 Principal Minor: A principal minor of a square matrix A is a minor of A where the indices of the deleted rows are the same as the indices of the deleted columns.

Ex: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ and take $k_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}, k_2 = b_2$ then k_1 is the principal minor of A obtained by deleting 1^{st} row and 1^{st} column of A, k_2 is the principal minor of A obtained by deleting $1^{st}, 2^{nd}$ row and $1^{st}, 2^{nd}$ column of A. Note that minor $k = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ is not the principal minor of A since it is obtained by deleting 1^{st} row while 2^{nd} column.

1.6.4 Leading principal minor: The leading principal minor of order k of an $n \times n$ matrix A is obtained by deleting the last $n - k$ rows and columns of the matrix.

Ex: Let $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$ then $k_1 = \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}, k_2 = \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \& k_3 = a_1$ are the principal minors of A, k_1 is obtained by deleting zero rows and zero columns of A, k_2 is obtained by deleting last one row and column, k_3 is obtained by deleting last two rows and columns.

1.6.5 Cofactor: The cofactor corresponding to element a_{ij} of square matrix $A = [a_{ij}]_{n \times n}$ is $A_{ij} = (-1)^{i+j} M_{ij}$, where M_{ij} is the minor of A obtained by deleting i^{th} row and j^{th} column of A.

Ex: Co-factors of matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \\ 3 & 1 & 2 \end{bmatrix}$ are given as:

$$A_{11} = \begin{vmatrix} 1 & 3 \\ 1 & 2 \end{vmatrix} = -1, \quad A_{12} = (-1)\begin{vmatrix} 2 & 3 \\ 3 & 2 \end{vmatrix} = 5, \quad A_{13} = \begin{vmatrix} 2 & 1 \\ 3 & 1 \end{vmatrix} = -1$$

$$A_{21} = (-1)\begin{vmatrix} 2 & 3 \\ 1 & 2 \end{vmatrix} = 1, \quad A_{22} = \begin{vmatrix} 1 & 3 \\ 3 & 2 \end{vmatrix} = -7, \quad A_{23} = (-1)\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = 5$$

$A_{31} = \begin{vmatrix} 2 & 3 \\ 1 & 3 \end{vmatrix} = 3,$

$A_{32} = (-1)\begin{vmatrix} 1 & 3 \\ 2 & 3 \end{vmatrix} = 3, \qquad A_{33} = \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = -3$

1.6.6 Properties of determinant:

- Determinant of a square matrix is unchanged if we multiply by some scalar in any row (or column) and then add it to some other row (or column).
  Ex: Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $B = \begin{bmatrix} a + kc & b + kd \\ c & d \end{bmatrix}$. Find that $\det(A) = \det(B)$.
- Determinant of a square matrix A changes its sign if we interchange any two rows (or columns) of A.
- Determinant of a square matrix A is multiplied by scalar k if we multiply any row (or column) of A by some scalar k.
  Note:- We will discuss above three properties in detail when we talk about elementary matrices.
- Determinant of a matrix is zero if it has a zero row (or column).
- If two rows (or columns) of some square matrix A are identical, then $\det(A) = 0$.
- The value of determinant of any matrix A is remains unchanged if its rows are changed into columns the columns into rows. That is $\det(A) = \det(A^T)$.
- If each element of a row (or column) of a determinant is a some of few terms then the determinant can be split accordingly i.e., $\begin{vmatrix} \sum_{i=1}^{n} a_i & \sum_{i=1}^{n} b_i \\ c & d \end{vmatrix} = \sum_{i=1}^{n} \begin{vmatrix} a_i & b_i \\ c & d \end{vmatrix}$, particularly $\begin{vmatrix} a_1 + a_2 & b_1 + b_2 \\ c & d \end{vmatrix} = \begin{vmatrix} a_1 & b_1 \\ c & d \end{vmatrix} + \begin{vmatrix} a_2 & b_2 \\ c & d \end{vmatrix}$.
- Let $D(x) = \begin{vmatrix} p_1(x) & p_2(x) \\ p_3(x) & p_4(x) \end{vmatrix}$ then $\lim_{x \to a} D(x) = \begin{vmatrix} \lim_{x \to a} p_1(x) & \lim_{x \to a} p_2(x) \\ \lim_{x \to a} p_3(x) & \lim_{x \to a} p_4(x) \end{vmatrix}$, provided $\lim_{x \to a} p_i(x)$ exists finitely for each $i = 1,2,3,4$.
- Let $D(x) = \begin{vmatrix} p_1(x) & p_2(x) \\ p_3(x) & p_4(x) \end{vmatrix}$, then
$$\frac{d}{dx} D(x) = \begin{vmatrix} \frac{d}{dx} p_1(x) & \frac{d}{dx} p_2(x) \\ p_3(x) & p_4(x) \end{vmatrix} + \begin{vmatrix} p_1(x) & p_2(x) \\ \frac{d}{dx} p_3(x) & \frac{d}{dx} p_4(x) \end{vmatrix}$$
- Let $D(x) = \begin{vmatrix} p_1(x) & p_2(x) \\ m & n \end{vmatrix}$, where m, n are constants then,
$$\int_a^b D(x)dx = \begin{vmatrix} \int_a^b p_1(x)dx & \int_a^b p_2(x)dx \\ m & n \end{vmatrix}$$

Note: All the properties above are valid for general $n \times n$ matrices.

1.6.7 Proposition 1: Let A and B are square matrices of order n and k is any scalar then

- $|A^\theta| = |A|$
- $|kA| = k^n |A|$
- $|AB| = |BA|$
- $|A^m| = |A|^m = |A| \cdot |A| \dots |A| \{m - times\}$

1.6.7 Proposition 2: Determinant of idempotent matrices are either 0 or 1.
Let A is an idempotent matrix so $A^2 = A$.

- $|A^2| = |A|$
- $|A|^2 = |A|$
- $|A|^2 - |A| = 0$
- $|A|(|A| - 1) = 0$
- $|A| = 0 \; or \; |A| = 1$

1.6.8 Proposition 3: Determinant of involutory matrices are either 1 or -1.
Let A is an involutory matrix so $A^2 = I$.

- $|A^2| = |I| = 1$
- $|A|^2 = 1$
- $|A| = \pm 1$

1.6.9 Proposition 4: Determinant of nilpotent matrices are 0.
Let A is a nilpotent matrix so $A^k = \mathbf{0}$, for some $k \in \mathbb{N}$.

- $|A^k| = |\mathbf{0}| = 0$
- $|A|^k = 0$
- $|A| = 0$

1.6.10 Proposition 5: Determinant of orthogonal matrices are either 1 or – 1.
Let A is a orthogonal matrix so $A\,A^T = I$.

- $|AA^T| = |I| = 1$
- $|A||A^T| = 1$
- $|A||A| = 1 \qquad \{\because |A| = |A^T|\}$
- $|A|^2 = 1$
- $|A| = \pm 1$

1.6.11 Proposition 5: Determinant of skew-symmetric and skew-Hermitian matrices of odd order is equal to zero.
Let A is a skew-symmetric matrix of order n so $A^T = -A$.

- $|A^T| = |-A|$
- $|A| = (-1)^n |A|$
- $|A| = -|A| \qquad \{\because n \; is \; odd\}$
- $|A| = 0$

Similarly we can prove for skew-Hermitian matrices of odd order.

1.6.12 Adjoint of a square matrix: Transpose of cofactor matrix $A = \left[a_{ij}\right]_{n \times n}$ is known a adjoint of the matrix. It is denoted by $adj\ (A)$. Cofactor matrix is a $n \times n$ matrix whose entries are the cofactors of A. i.e., if $A = \left[a_{ij}\right]_{n \times n}$ then,

$adj(A) = [A_{ji}]_{n\times n}$, where A_{ij} is the cofactor of A corresponding to a_{ij}.

Ex: Let $A = \begin{bmatrix}1 & 2 & 3\\ 2 & 1 & 3\\ 3 & 1 & 2\end{bmatrix}$ then cofactors of A are given as:

$A_{11} = \begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix} = -1$, $A_{12} = (-1)\begin{vmatrix}2 & 3\\ 3 & 2\end{vmatrix} = 5$, $A_{13} = \begin{vmatrix}2 & 1\\ 3 & 1\end{vmatrix} = -1$

$A_{21} = (-1)\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix} = 1$, $A_{22} = \begin{vmatrix}1 & 3\\ 3 & 2\end{vmatrix} = -7$, $A_{23} = (-1)\begin{vmatrix}1 & 2\\ 3 & 1\end{vmatrix} = 5$

$A_{31} = \begin{vmatrix}2 & 3\\ 1 & 3\end{vmatrix} = 3$, $A_{32} = (-1)\begin{vmatrix}1 & 3\\ 2 & 3\end{vmatrix} = 3$, $A_{33} = \begin{vmatrix}1 & 2\\ 2 & 1\end{vmatrix} = -3$

Let's denote cofactor matrix of A by B then $B = \begin{bmatrix}A_{11} & A_{12} & A_{13}\\ A_{21} & A_{22} & A_{23}\\ A_{31} & A_{32} & A_{33}\end{bmatrix}$ and so $(A) = \begin{bmatrix}A_{11} & A_{21} & A_{31}\\ A_{12} & A_{22} & A_{32}\\ A_{13} & A_{23} & A_{33}\end{bmatrix} = \begin{bmatrix}-1 & 1 & 3\\ 5 & -7 & 3\\ -1 & 5 & -3\end{bmatrix}$.

1.6.13 Theorem: Matrices A and $adj(A)$ are commutative and their product is a scalar matrix with each diagonal entry $|A|$. i.e., $A \cdot adj(A) = adj(A) \cdot A = |A|I_n$.

Proof: Let $A = [a_{ij}]_{n\times n}$ is a matrix of order n and $adj(A) = [A_{ji}]_{n\times n}$, where A_{ij} is the cofactor of A corresponding to a_{ij}.

Now $(i,j)^{th}$ entry of $A \cdot adj(A) = \sum_{k=1}^{n} a_{ik}A_{jk} = \begin{Bmatrix}0\,,\ if\ i \neq j\\ |A|\,,\ if\ i = j\end{Bmatrix} = adj(A) \cdot A$. Therefore $A \cdot adj(A) = adj(A) \cdot A = |A|I_n$.

Remark: If A is invertible matrix then from $A \cdot adj(A) = |A|I_n$ we have $A^{-1} = \frac{adj(A)}{|A|}$. So A is invertible $|A| \neq 0$.

1.6.14 Properties of inverse of a square matrix: If A and B are invertible matrices of order n and $k \neq 0$ is any number then

- $(A^{-1})^{-1} = A$
- $(kA)^{-1} = k^{-1}A^{-1}$
- $(AB)^{-1} = B^{-1}A^{-1}$
- $(A^T)^{-1} = (A^{-1})^T$
- $\left(A^{\theta}\right)^{-1} = (A^{-1})^{\theta}$
- A triangular matrix is invertible iff all its diagonal element's are non-zero.
- Involutary matrices are invertible and they are self inverse i.e. if A is involutory then $A^{-1} = A$.
- No nilpotent matrix is invertible since nilpotent matrices are singular.
- Every orthogonal matrix is invertible and their inverse is its transpose. Let A is orthogonal then $AA^T = I \Rightarrow A^{-1} = A^T$.
- Every unitary matrix is invertible and their inverse is their conjugate transpose. Let A is unitary then $AA^{\theta} = I \Rightarrow A^{-1} = A^{\theta}$

  We can easily verify all the above properties with the help of $A^{-1} = \frac{adj(A)}{|A|}$.

1.6.15 Corollary 1:- Let A is $n \times n$ square matrix then $|adj(A)| = |A|^{n-1}$.

From above theorem we have $A \cdot adj(A) = |A|I_n$.

$\Rightarrow |A \cdot adj(A)| = \big||A|I_n\big|$

$\Rightarrow |A| \cdot |adj(A)| = |A|^n|I_n| = |A|^n$

$\Rightarrow |adj(A)| = |A|^{n-1}$

1.6.16 Corollary 2: $adj(AB) = adj(B) \cdot adj(A)$

We have $A \cdot adj(A) = |A|I_n$

$\Rightarrow adj(A) = |A|A^{-1}$

$\Rightarrow adj(AB) = |AB|(AB)^{-1} = |A||B|B^{-1}A^{-1} = (|B|B^{-1})(|]A|A^{-1})$

1.6.17 Corollary 3: $adj(kA) = k^{n-1}adj(A)$ for any scalar k.

We have $adj(A) = |A|A^{-1}$.

$\Rightarrow adj(kA) = |kA|(kA)^{-1} = k^n|A|k^{-1}A^{-1}$

$\Rightarrow adj(kA) = k^{n-1}|A|A^{-1}$

$\Rightarrow adj(kA) = k^{n-1}adj(A)$

1.6.18 Corollary 4: $adj\big(adj(A)\big) = |A|^{n-2}I_n$, provided A is invertible.

We have $A \cdot adj(A) = |A|I_n$. Take adj both side.

$\Rightarrow adj\big(A \cdot adj(A)\big) = adj(|A|I_n)$

$\Rightarrow adj\big(adj(A)\big) \cdot adj(A) = |A|^{n-1}adj(I_n) = |A|^{n-1}I_n$

$\Rightarrow adj\big(adj(A)\big) = |A|^{n-1}I_n(adj(A)^{-1}) = |A|^{n-1}|A|^{-1}A$ $\{\because adj(A) = |A|A^{-1}\}$

$\Rightarrow adj\big(adj(A)\big) = |A|^{n-2}A$.

1.6.19 Corollary 5: If A is square matrix of order n then $|adj\ adj\ \ldots.adj\ A| = |A|^{(n-1)k}$($k$- times). Since $|adj\ (A)| = |A|^{n-1}$

$\Rightarrow |adj\ adj\ (A)| = |adj\ A|^{n-1}\ \{Relapce\ A\ with\ adj(A)\}$

$\Rightarrow |adj\ adj\ (A)| = (|A|^{n-1})^{n-1} = |A|^{(n-1)^2}$

Now $|adj\ adj\ adj\ (A)| = |adj\ A|^{(n-1)^2}$

$\{Again\ relplace\ A\ with\ adj(A)\ in\ the\ above\ equation\}$

$\Rightarrow |adj\ adj\ adj\ (A)| = (|A|^{n-1})^{(n-1)^2} = |A|^{(n-1)^3}$

Keep doing this upto k- times we will get $|adj\ adj\ \ldots adj\ A| = (|A|)^{(n-1)^k}$.

Q. If A is a square matrix of order n and $|A| = 3$, then which of the following is/are correct?

(a) $|adj\ (A)| = 3^n$
(b) $|adj(A)| = 3$
(c) $|adj\ adj\ adj\ (A)| = 3^{(n-1)^3}$
(d) $|adj\ adj\ (A)| = 3^{(n-1)^3}$
Solution: We have $|adj(A)| = |A|^{n-1} = 3^{n-1}$ so option (a) and (b) are not correct.
Now $|adj\ adj\ ....adj\ (A)| = |A|^{(n-1)k}$($k$- times)
$\Rightarrow |adj\ adj\ adj\ (A)| = 3^{(n-1)^3}$
Hence only option (c) is correct.

Q. Let $A = \begin{bmatrix} \frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{1}{3} & -\frac{2}{3} \\ \frac{-2}{3} & \frac{2}{3} & -\frac{1}{3} \end{bmatrix}$ then A^{-1} =?
(a) A^T (b) A
(c) $2A$ (d) None.
Solution: Observe that A is an orthogonal matrix so $A^{-1} = A^T$. Hence option A is correct.

Q. The determinant of the $n \times n$ permutation matrix $\begin{bmatrix} 0 & 0 & 0 & \cdots & 1 \\ 0 & 0 & \cdots & 1 & 0 \\ 0 & \cdots & \cdot & 0 & 0 \\ 0 & \cdot & \cdots & 0 & 0 \\ 1 & \cdots & 0 & 0 & 0 \end{bmatrix}$
(a) $(-1)^n$ (b) $(-1)^{\left\lfloor \frac{n}{2} \right\rfloor}$
(c) -1 (d) 1
Here $\lfloor x \rfloor$ denotes the greatest integer not exceeding x.
Solution: (b)
For $n = 1 \Rightarrow A = [1]_{1\times1} \Rightarrow \det A = 1$
So, option (a) and (c) are incorrect.
For $n = 2, A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \Rightarrow \det A = -1$
So, option (d) is incorrect
Thus option (b) is correct.

Q. The determinant $\begin{vmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{vmatrix}$ is equal to
(a) $(z-y)(z-x)(y-x)$
(b) $(x-y)(x-z)(y-z)$
(c) $(x-y)^2(y-z)^2(z-x)^2$
(d) $(x^2-y^2)(y^2-z^2)(z^2-x^2)$
Solution: (a)
A=$\begin{bmatrix} 1 & 1+x & 1+x+x^2 \\ 1 & 1+y & 1+y+y^2 \\ 1 & 1+z & 1+z+z^2 \end{bmatrix}$
$R_2 \to R_2 - R_1, R_3 \to R_3 - R_1,$
$\sim \begin{bmatrix} 1 & 1+x & 1+x+x^2 \\ 0 & y-x & (y-x)+(y^2-x^2) \\ 0 & z-x & (z-x)+(z^2-x^2) \end{bmatrix}$
$\Rightarrow |A| = (y-x)\{(z-x)+(z^2-x^2)\} - (z-x)\{(y-x)+(y^2-x^2)\} + 0 + 0$
$i.e., |A| = (y-x)(z-x)[1+z+x-1-y-x]$
$i.e., |A| = (y-x)(z-x)(z-y)$

Q. Let $D_1 = \det\begin{pmatrix} a & b & c \\ x & y & z \\ p & q & r \end{pmatrix}$ and $D_2 = \det\begin{pmatrix} -x & a & -p \\ y & -b & q \\ z & -c & r \end{pmatrix}$.
Then
(a) $D_1 = D_2$ (b) $D_1 = 2D_2$
(c) $D_1 = -D_2$ (d) $2D_1 = D_2$
Solution: (c)
$D_1 = \det\begin{bmatrix} a & b & c \\ x & y & z \\ p & q & r \end{bmatrix}$
Take minus common from first row and then from second column.
$D_1 = (-)(-)\begin{vmatrix} -a & b & -c \\ x & -y & z \\ p & -q & r \end{vmatrix} = D_2$
Hence, option (c) is correct.
30. For $t \in \mathbb{R}$, define
$M(t) = \begin{pmatrix} 1 & t & 0 \\ 1 & 1 & t^2 \\ 0 & 1 & 1 \end{pmatrix}$

Q. Then which of the following statements is true?
(a) det M (t) is a polynomial function of degree 3 in t
(b) det M (t) = 0 for all $t \in \mathbb{R}$
(c) det M (t) is zero for infinity many $t \in \mathbb{R}$
(d) det M (t) is zero for exactly two $t \in \mathbb{R}$
Solution: (d)
$\det M(t) = 1 - t - t^2$.
Note that $b^2 - 4ac = 5 > 0$.
Thus, option (d) is correct.

Q. Let $A = \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}$ be a matrix. Which of the following is true?
(a) $A^* = A^{-1}$ (b) $AA^* = A^*A$
(c) $A^* = A$ (d) $A^2 = I$
Solution: (b)
$A = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}$
$A^* = \left(\overline{A}\right)^T$
$\Rightarrow A^* = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}^T = \begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$.....(i)
So, option (c) is incorrect.
Now $A^{-1} = \frac{1}{2}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$..... (ii)
∴ Option (a) is incorrect
$A^2 = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2i \\ 2i & 0 \end{bmatrix} \neq I_{2\times2}$
∴ Option (d) is incorrect
Now, $A.A^* = \begin{bmatrix} 1 & i \\ i & 1 \end{bmatrix}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = A^*A$
Hence, option (b) is correct.

Q. Let A and B be 2X2 matrices. Then which of the following is true?

(a) $\det(A+B)+\det(A-B) = \det A + \det B$

(b) $\det(A+B)+\det(A-B) = 2\det A - 2\det B$

(c) $\det(A+B)+\det(A-B) = 2\det A + 2\det B$

(d) $\det(A+B)-\det(A-B) = 2\det A - 2\det B$

Solution: (c)

Let

$A = \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}, B = \begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$

$\det(A) = (a_1a_4 - a_2a_3)$, $\det(B) = b_1b_4 - b_2b_3$.

Now, $A+B = \begin{bmatrix} a_1+b_1 & a_2+b_2 \\ a_3+b_3 & a_4+b_4 \end{bmatrix}$ and

$A-B = \begin{bmatrix} a_1-b_1 & a_2-b_2 \\ a_3-b_3 & a_4-b_4 \end{bmatrix}$.

$\Rightarrow \det(A+B) = (a_1+b_1)(a_4+b_4) - (a_2+b_2)(a_3+b_3)$

$= a_1a_4 + a_1b_4 + a_4b_1 + b_1b_4 - a_2a_3 - a_2b_3 - b_2a_3 - b_2b_3$

$\det(A-B) = (a_1-b_1)(a_4-b_4) - (a_2-b_2)(a_3-b_3)$

$= a_1a_4 - a_1b_4 - a_4b_1 + b_1b_4 - a_2a_3 + a_2b_3 + b_2a_3 - b_2b_3$

So, $\det(A+B) + \det(A-B) = 2(a_1a_4 + b_1b_4) - 2(a_2a_3 - b_2b_4)$

$= 2[(a_1a_4 - a_2a_3) + (b_1b_4 - b_2b_3)]$

$\Rightarrow \det(A+B) + \det(A-B) = 2[\det(A) + \det(B)]$

Or

Consider $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$, and discard options (a), (b) and (d).

Hence option (c) is correct.

Q. Let A and B be $n \times n$ matrices. Suppose the sum of the elements in any row of A is 2 and the sum of the elements in any column of B is 2. Which of the following matrices is necessarily singular?

(a) $I - \frac{1}{2}BA^T$ (b) $I - \frac{1}{2}AB$

(c) $I - \frac{1}{4}AB$ (d) $I - \frac{1}{4}BA^T$

Solution: (d)

Let $A = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}$

$AB = \begin{bmatrix} -1 & 3 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 10 & 2 \\ 2 & 2 \end{bmatrix}$

$BA^T = \begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} -1 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$

$\Rightarrow$ Options (a), (b) and (c) are incorrect.

Thus, option (d) is correct

1.7 RANK OF MATRIX

1.7.1 Definition: Let $A = [a_{ij}]_{m\times n}$ be a matrix. A positive integer r is said to be the rank of matrix A if

(i) Matrix A has at least r- rowed non zero minor

(ii) Every $(r+1)$ row minor should be zero.

We denote it by $rank(A)$ or $\rho(A)$.

1.7.2 Properties:

- Rank of non-singular matrix A of order n is n because n-rowed minor is $|A|$ which is non zero.
- Rank of singular matrix of order n is less than n since we can not have any non-zero minor of size n.
- Rank of null-matrix is zero since we can not have any non zero minor of null(or zero) matrix.
- Rank of non-null matrix are always greater than or equal to 1, since we can always have at least one non zero minor of size 1 or grater.
- If A is a $m \times n$ matrix then $rank(A) \leq \min\{m,n\}$ because we can have non zero minor of size at most $\min\{m,n\}$.

Note: We will see some more properties of rank when we will talk about L.I and L.D vectors.

Ex: Find the rank of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$.

Solution: We have $|A| = 1(3\times 7 - 4\times 5) - 2(2\times 7 - 3\times 4) + 3(2\times 5 - 3\times 3) = 1 - 4 + 3 = 0$. So we can not have any non-zero minor of size 3 i.e., rank of A will be less than 3. Let's take minor $\begin{vmatrix} 2 & 3 \\ 3 & 4 \end{vmatrix} = 8 - 9 = -1 \neq 0$. Note that we have a non-zero minor of size 2, therefore rank of A is 2.

1.7.3 Elementary transformations:

The elementary transformations of a matrix are the operations performed on rows and columns of a matrix to transform the given matrix into a different form in order to make the calculation simpler. Elementary row operations are the operations that are performed on rows of a matrix. Similarly, elementary column operations are the operations that are performed on columns of a matrix.

The three basic elementary transformations of a matrix are:

1. **Interchange of any two rows or two columns:** If we interchange i^{th} row (column) with j^{th} row (column), we denote it by $R_i \leftrightarrow R_j$ $(C_i \leftrightarrow C_j)$.
2. **Multiplication of row or column by a non-zero number (scalar):** If we multiply i^{th} row (column) by scalar k, we denote it by $R_i \to kR_i (C_i \to kC_i)$.
3. **Multiplication of row or column by a non-zero number and add the result to the other row or column:** If we multiply j^{th} row (column) by scalar k and then adding it to i^{th} row (column), we denote it by $R_i \to R_i + kR_j (C_i \to C_i + kC_j)$.

Ex: Consider $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, let's apply above elementary transformations on A.

1. $R_1 \leftrightarrow R_2 \Rightarrow A \sim \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$
2. $R_3 \to kR_3 \Rightarrow A \sim \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ ka_3 & kb_3 & kc_3 \end{bmatrix}$
3. $R_1 \to R_1 + kR_2 \Rightarrow A \sim \begin{bmatrix} a_1 + ka_2 & b_1 + kb_2 & c_1 + kc_2 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$

1.7.4 Elementary Matrices: A matrix obtained by a single elementary transformation on identity matrix is called an elementary matrix. Note that $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ is an elementary matrix obtained by elementary row transformation $R_1 \leftrightarrow R_2$ on I_3. We have three types of elementary matrices corresponding to three types of elementary transformations.

1. E_{ij} – corresponding to $R_i \leftrightarrow R_j$ $(C_i \leftrightarrow C_j)$
2. $E_i(\lambda)$ – corresponding to $R_i \to \lambda R_i$ $(C_i \to \lambda C_i)$
3. $E_{ij}(\lambda)$ – corresponding to $R_i \to R_i + \lambda R_j$ $\left(C_i \to C_i + \lambda C_j\right)$

Note: Every elementary row (or column) transformation of a matrix can be obtained by pre multiplication (or post multiplication) with the corresponding elementary matrix.

Ex: Consider $A = \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}$, let's we want to apply row transformation $R_1 \leftrightarrow R_2$ to A. Find corresponding elementary matrix, which is $E_{12} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$. Pre-multiply E_{12} with A we will get the desired row transformation on A, as $E_{12}A = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix} = \begin{bmatrix} a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{bmatrix}$.

1.7.5 Inverse of Elementary matrices: With the help of above note we can easily find the inverse of elementary matrices as:

(1) $E_{ji} \cdot E_{ij} = I \Rightarrow \left(E_{ij}\right)^{-1} = E_{ji}$

(2) $E_i\left(\frac{1}{k}\right) \cdot E_i(k) = I \Rightarrow \left(E_i(k)\right)^{-1} = E_i\left(\frac{1}{k}\right)$

(3) $E_{ij}(-k) \cdot E_{ij}(k) = I \Rightarrow \left(E_{ij}(k)\right)^{-1} = E_{ij}(-k)$

Ex: Find the inverse of $A = \begin{bmatrix} 1 & 0 & 0 & 5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

Solution: Observe that A is nothing but $E_{14}(5)$ and inverse of $E_{14}(5)$ is $E_{14}(-5)$. Therefore $A^{-1} = E_{14}(-5) = \begin{bmatrix} 1 & 0 & 0 & -5 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$.

1.7.6 Equivalence of matrices: Let A is a $m \times n$ matrix, a $m \times n$ matrix B obtained from matrix A by finite number of elementary transformations on A is called equivalent to A. If A is equivalent to B be denote it by $A \sim B$.

Ex: Every elementary matrix is equivalent to identity matrix.

1.7.7 Row-reduced form of a Matrix: An $m \times n$ matrix A is called row-reduced if:

(a) the first non-zero entry in each non-zero row of A is equal to 1;

(b) each column of A which contains the leading non-zero entry of some row has all its other entries 0.

Ex: One example of a row-reduced matrix is the identity matrix I. Two examples of matrices which are not row-reduced are : $\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 2 & 1 \\ 1 & 0 & -3 \\ 0 & 0 & 0 \end{bmatrix}$

Note that second matrix fails to satisfy condition (a), because the leading non-zero entry of the first row is not 1. The first matrix does satisfy condition (a), but fails to satisfy condition (b) in column 3.

Note: Every $m \times n$ matrix is row equivalent to a row reduced matrix. Row equivalent is in the sense, we find matrix B from A by only elementary row transformations.

1.7.8 Row-reduced echelon for of matrix: An $m \times n$ rnatrix A is called a row-reduced echelon form if:

- A is row-reduced;
- every row of R which has all its entries 0 occurs below every row which has a non-zero entry;
- if rows $1,2 \ldots, r$ are the non-zero rows of A, and if the leading non-zero entry of row i occurs in column k_i, $i = 1,2 \ldots, r$, then $k_1 < k_2 < \ldots < k_r$.

or

An $m \times n$ rnatrix A is called a row-reduced echelon form if:

- Every row of A with each zero entry occurs below every row which has a non-zero entry.
- The number of zeros before the first non-zero element in a row is less than the number of such zeros in the next row.

  The rank of a matrix in echelon form is equal to the number of non-zero rows of the matrix.

Note: In the second definition of Echelon form, we do not need 1 as the first non–zero entry in each row. Some authors don't require that the leading coefficient is 1 in Echelon form. Basically Echelon form is used to find the rank of a matrix, in finding rank by Echelon form there is no need to make 1 as first non-zero entry in each row.

Remark: Every $m \times n$ matrix A is row-equivalent to a row-reduced echelon matrix. i.e., every $m \times n$ matrix can be converted in row reduced Echelon form.

1.7.9 Pivot: The first non-zero entry in any row of a matrix in row reduced Echelon form is called pivot. Rank of any matrix A is equals to the number of pivots in it.

Q. Find the Echelon form of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$ and then find the rank it.

Solution: $R_{2\rightarrow}R_2 - 2R_1$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \end{bmatrix}$$

$$R_{3\rightarrow}R_3 - 3R_1$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & -1 & -2 \end{bmatrix}$$

$$R_{3\rightarrow}R_3 - R_2$$

$$\sim \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

$$R_{3\rightarrow}R_3 \times (-1)$$

$$\sim \begin{bmatrix} \textcircled{1} & 2 & 3 \\ 0 & \textcircled{1} & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Echelon form of the matrix is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Find that, in Echelon form of A, 2 non-zero rows are there so $rank(A) = 2$. We can also note that there are two pivots(inside circle) so $rank(A) = 2$.

(Multiple Choice Questions) MCQ

Q. Let D be a non-zero $n \times n$ real matrix with $n \geq 2$. Which of the following implications is valid?

(a) $\det(D) = 0$ implies rank $(D) = 0$

(b) $\det(D) = 1$ implies rank $(D) \neq 1$

(c) rank $(D) = 1$ implies $\det(D) \neq 0$

(d) rank $(D) = n$ implies $\det(D) \neq 1$

Solution: (b)

Given, D be a non-zero $n \times n$ real matrix with ≥ 2 .

For option (a) and (c) , let D=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

$\Rightarrow \det(D) = 0$

$\Rightarrow$rank (D) $= 1$

Hence option a and c are incorrect.

For option (d), let D=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

Here n=2 and rank(D)=2

But $\det(D) = 1$

Hence option d is incorrect.

For option (b), $\det(D) \neq 0 \Rightarrow rank(D) = n$.

Hence option (b) is true.

Q. The determinant of the matrix

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$ is ?

(a) 0 (b) – 9

(c) – 27 (d) 1

Solution: (c)

We have A = $\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 2 & 1 & 0 & 0 \\ 0 & 2 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$

We will apply row operations , $R_6 \rightarrow R_6 - 2R_1$, $R_5 \rightarrow R_5 - 2R_2$, $R_4 \rightarrow R_4 - 2R_3$ which gives

$$\sim \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 2 & 0 & 0 \\ 0 & 0 & 0 & -3 & 0 & 0 \\ 0 & 0 & 0 & 0 & -3 & 0 \\ 0 & 0 & 0 & 0 & 0 & -3 \end{bmatrix}$$

Now $\det(A) = 1 \times 1 \times 1 \times -3 \times -3 \times -3 = -27$.

Q. Let A, B *be* $n \times n$ real matrices. Which of the following statements is correct?

(a) Rank $(A + B) = rank(A) + rank\ (B)$

(b) Rank $(A + B) \leq rank(A) + rank\ (B)$

(c) Rank $(A + B) = \min\{rank(A), rank\ (B)\}$

(d) Rank $(A + B) = \max\{rank(A), rank\ (B)\}$

Solution: (b)

Let us take $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$; Rank$(A) = 1$ and

$B = \begin{bmatrix} -1 & 0 \\ 0 & 0 \end{bmatrix}$; $Rank(B) = 2$

Then $A + B = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$\Rightarrow$ Rank $(A + B) = 0 \neq Rank(A) + Rank(B)$

Now

$\min\{Rank(A), Rank(B)\} = 1$ and

$\max\{Rank(A), Rank(B)\} = 1$

$\Rightarrow$ Rank $(A + B) \neq \min\{Rank(A), Rank(B)\}$ and Rank $(A + B) \neq \max\{Rank(A), Rank(B)\}$

Hence option (a), (c) and (d) are incorrect .

Hence option (b) is correct.

Q. Let $A_{n\times n} = \left((a_{ij})\right), n \geq 3$, where $a_{ij} = \left(b_i^2 - b_j^2\right), i, j = 1,2, \dots, n$ for some distinct real numbers $b_1, b_2, \dots, b_n$. Then $\det(A)$ is

(a) $\prod_{i<j}(b_i - b_j)$ (b) $\prod_{i<j}(b_i + b_j)$

(c) 0 (d) 1

Solution: (c)

For n=3,

A=$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 0 & b_1^2 - b_2^2 & b_1^2 - b_3^2 \\ b_2^2 - b_1^2 & 0 & b_2^2 - b_3^2 \\ b_3^2 - b_1^2 & b_3^2 - b_2^2 & 0 \end{bmatrix}$

Note that A is a skew symmetric matrix and order of A is 3. We know that determinant of a skew symmetric matrix of odd order is zero.

$\Rightarrow \det(A) = 0$.

Hence option (c) is correct.

Q. Let J denote the matrix of order $n \times n$ with all entries 1 and let B be a $(3n) \times (3n)$ matrix given by $B = \begin{pmatrix} O & O & J \\ O & J & O \\ J & O & O \end{pmatrix}$. Then the rank of B is

(a) $2n$ (b) $3n - 1$

(c) 2 (d) 3

Solution: (d)

Let $J = [1]$

Then $B = \begin{bmatrix} 0 & 0 & J \\ 0 & J & 0 \\ J & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

$\rho(B) = 3$

option (a), (b) and (c) are incorrect.

Thus option (d) is correct.

Q. Let A be an $n \times m$ matrix with each entry equal to $+1, -1$ or 0 such that every column has exactly one +1 and exactly one – 1 . We can conclude that

(a) Rank $A \leq n - 1$ (b) Rank $A = m$

(c) $n \leq m$ (d) $n - 1 \leq m$

Solution: (a) $A = (a_{ij})$, is $m \times n$ matrix

Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \\ -1 & -1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}_{5x2}$

Here $\mathrm{n} = 5, \mathrm{m} = 2$.

Note that $rank(A) = 1 \neq m$.

Therefore option (b),(c),(d) are incorrect.

Hence option (a) is correct.

Q. Let A be $(m \times n)$ matrix and B be a $(n \times m)$matrix over real numbers with $m < n$. Then

(a) AB is always non-singular (b) AB is always singular

(c) BA is always non-singular (d) BA is always singular

Solution: (d)

$[A]_{m \times n}$, $[B]_{n \times m}$,m<n

Take m=2,n=3

$i.e., m < n$

(a) Let A=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,B=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

AB=$\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

$\Rightarrow |AB| = 0$

$\Rightarrow singular$

hence (a) is incorrect.

(b) A=$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$,B=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$, AB=$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$|AB|=1≠0

$\Rightarrow$non singular

Hence b is incorrect.

(c) A=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$,B=$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$

$\Rightarrow$BA=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$

|BA|=0

Hence (c) is incorrect.

Hence option (d) is correct.

Q. What is the rank of the following matrix?

$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

(a) 2 (b) 3

(c) 4 (d) 5

Solution:(d)

The given matrix is, $A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 2 & 2 & 2 \\ 1 & 2 & 3 & 3 & 3 \\ 1 & 2 & 3 & 4 & 4 \\ 1 & 2 & 3 & 4 & 5 \end{pmatrix}$

Use simple row operations to get $rank(A) = 5$.

Q. Let a, b and c be distinct integers. Let A be the matrix

$A = \begin{pmatrix} a^2 & b^2 & c^2 \\ a^5 & b^5 & c^5 \\ a^{11} & b^{11} & c^{11} \end{pmatrix}$

Which of the following is the set of all possible ranks of A?

(a) {3} (b) {2,3}

(c) {1,2,3} (d) {0,1,2,3}

Solution: (b)

Case 1- Suppose rank (A) = 0

That means, A is zero matrix.

$A = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

Hence $a = 0, \quad b = 0, \quad c = 0$

But a, b and c are distinct integers Therefore, our assumption rank (A) = 0 is not possible. Hence (d) is wrong.

Case 2 - Suppose rank (A) = 1, that means all the minors of order 2 are zeros.

WLOG suppose a & b are distinct nonzero integers

$A = \begin{bmatrix} a^2 & b^2 & c^2 \\ a^5 & b^5 & c^5 \\ a^{11} & b^{11} & c^{11} \end{bmatrix}$

Minor M_{33}=0

$i.e., \begin{vmatrix} a^2 & b^2 \\ a^5 & b^5 \end{vmatrix} = 0$

$i.e., a^2b^5 - a^5b^2 = 0$

$i.e., b^3 - a^3 = 0$

$\Rightarrow b = a$

But a & b are distinct integers.

Hence rank 1 is not possible.

So (c) is incorrect.

Case 3 - Example : $a = -1$, $b = 1$, $c = 0$

$A = \begin{bmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$

Note that $\det(A) = 0$

So, rank (A) < 3

Minor $M_{33} = \left[\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}\right] = 2 \neq 0$

Therefore, rank (A)=2

Case 4 $a = -1$, $b = 1$, $c = 2$

$A = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 1 & 32 \\ -1 & 1 & 2048 \end{bmatrix}$

$\det(A)$ = 2016 - (- 2016) + 0

$= 4032 \neq 0$

Therefore , rank (A) = 3

Hence (b) is correct.

Q. Let M be a 5×5 matrix with real entries such that Rank(M)=3. Consider the linear system $Mx = b$. Let the row reduced echelon form of the augmented matrix $[M \ b]$ be R and let $R[i, :]$ denote the i-th row of R. Suppose that the linear system admits a solution . Which of the following statements is necessarily true?

(a) R[3, :]=[0 1 0 * * *]

(b) R[5, :]=[0 0 1 0 * *]

(c) R[4, :]=[0 0 0 1 * *]

(d) R[4, :]=[0 0 0 0 0 0]

Solution: (d)

$[M]_{5\times5}$ rank (M) = 3

Row reduced echelon form of M,

$M \sim \begin{bmatrix} 1 & 0 & 0 & * & * \\ 0 & 1 & 0 & * & * \\ 0 & 0 & 1 & * & * \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

If the row reduced echelon form of the augmented matrix [M b] of linear system $Mx = 0$ is R and the linear system admits a solution then, Rank (M) = Rank ([M b]) = Rank (R) = 3

$R = \begin{bmatrix} 1 & 0 & 0 & * & * & * \\ 0 & 1 & 0 & * & * & * \\ 0 & 0 & 1 & * & * & * \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

For option (a) – R [3, :] = [0 0 1 * * *]

Hence option (a) is wrong

For option (b) – R[5, :] = [0 0 0 0 0 0]

Hence option (b) is also wrong.

For option (c) – R[4, :] = [0 0 0 0 0 0]

Hence (c) is wrong

For option (d) – R [4, :] = [0 0 0 0 0 0]

Hence correct.

Q. Consider the two statements given below:

I. There exists a matrix $N \in \mathbb{M}_4(\mathbb{R})$ such that $\{(1,1,1,-1),(1,-1,1,1)\}$ is a basis of Row(N) and $(1,2,1,4) \in Null(N)$

II. There exists a matrix $M \in \mathbb{M}_4(\mathbb{R})$ such that $\{(1,1,1,0)^T,(1,0,1,1)^T\}$ is a basis of $Col(M)$ and $(1,1,1,1)^T,(1,0,1,0)^T \in Null(M)$

Which of the following statements is true?

(a) Statement is FALSE and Statement II is TRUE,

(b) Statement is TRUE and Statement II is FALSE,

(c) Both Statement and Statement II are FALSE

(d) Both Statement I and Statement II are TRUE

Solution: (a)

Row space is orthogonal to null space.

So basis vectors of row space should be orthogonal to vectors in null space.

$\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} = 0$

$\begin{bmatrix} 1 & 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 1 \\ 4 \end{bmatrix} = 4 \neq 0$

$\Rightarrow$ Statement I is false

For Statement II:

Let $M = \begin{bmatrix} 1 & 1 & x_1 & y_1 \\ 1 & 0 & x_2 & y_2 \\ 1 & 1 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{bmatrix}$, then

$M \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & x_1 & y_1 \\ 1 & 0 & x_2 & y_2 \\ 1 & 1 & x_3 & y_3 \\ 0 & 1 & x_4 & y_4 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

$= 1. \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1. \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} + 1. \begin{bmatrix} x_1 \\ \vdots \\ x_4 \end{bmatrix} + 1. \begin{bmatrix} y_1 \\ \vdots \\ y_4 \end{bmatrix}$

Similarly,

$M \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 & 1 & x_1 & y_1 \\ 1 & 0 & \vdots & \vdots \\ 1 & 1 & \vdots & \vdots \\ 0 & 1 & x_4 & y_4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = 1. \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix} + 1. \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$

According to given condition,