P R E F A C E

The principal purpose of writing this book **"IIT JAM STATISTICS PYQ Solutions"** is to cater to the needs of the candidates aspiring for "IIT JAM STATISTICS" exam. It provides explanations of the previous year questions from the year 2005 to 2025. To help students, we have compiled this book in a user-friendly and well-structured manner. Book includes 13 chapters. First four chapters covers the Mathematics section and remaining chapters covers the Statistics section. Wherever required, particular chapter is further subdivided into units. This fine sorting of questions will help the students to prepare for each subtopic effectively.

The book will help the reader to understand and solve all the previous year questions. Regular practice is the key to success in any competitive exam, and this book allows you to hone your problem-solving skills. We have tried to keep the material as simple as possible with easy explanations. The practice of these questions will help the students not only for IIT JAM exam but also for other competitive exams like CUET, GATE, CSIR-NET, PhD entrances, etc.

A word or suggestion from your side may add another feather to the cap in the compilation of questions in this book. The author looks forward to the comments, suggestions, and criticism from the students. Constructive suggestions and user feedback would be highly appreciated, gratefully acknowledged, and suitably incorporated. We wish all the best to all aspirants who are preparing for these exams.

IIT JAM ENTRANCE EXAM DETAILS

Joint Admission Test (JAM) is an all-India admission test conducted jointly by the Indian Institute of Technology (IITs) and Indian Institute of Science, Bangalore (IISC) since 2004 to provide admissions to various postgraduate programs as follows:

1. M.Sc. (Four Semesters)
2. M.Sc. (Tech.)
3. Joint M.Sc.-Ph.D.
4. M.Sc. - Ph.D. Dual Degree

ELIGIBILITY CRITERIA

1. Candidates who have completed an undergraduate degree or currently studying in the final year of undergraduate programme are eligible to apply for JAM examination.
2. Candidates who qualify in JAM 2025 will be eligible to apply for admission to the Academic programmes of the Admitting Institutes subject to satisfying the Eligibility Requirements (ERs) and Minimum Educational Qualifications (MEQs).

ADMITTING INSTITUTES

IIT JAM Mathematical Statistics (MS) test score will be used by following IITs for admission to various post graduate programs.

S. No.	Institute	Academic Programme [Code]	JAM Test Paper Code	Minimum Educational Qualifi-cations (MEQs) / Essential Subject in Bachelor's Degree along with minimum Duration
1	IIT Bombay	M.Sc. in Statistics [1203]	MS	No Restrictions
2	IIT Bombay	M.Sc. in Operations Research [1214]	MS	No Restrictions
3	IIT Kanpur	M.Sc. in Statistics [1504]	MS	No Restrictions
4	IIT Palakkad	M.Sc. in Mathematics [2902]	MA, MS	At least four Mathematics courses in Bachelor's Degree
5	IIT Tirupati	M.Sc. in Mathematics & Statistics [3001]	MS	No Restrictions

EXAM PATTERN

The exam will be Computer Based Test (CBT). The duration of the examination will be of 3 hours. The medium for test will be only in English. There will be a total of 60 questions carrying a total of 100 marks. The entire paper will be divided into three sections A, B and C. All sections are compulsory. Questions in each section will be of different types as given below:

Sections	**Section A**	**Section B**	**Section C**
Number of Questions	30	10	20
Marks	10 ques. of 1 mark each and 20 ques. of 2 marks each.	2 marks for each question	10 ques. of 1 mark each and 10 ques. of 2 marks each.
Negative Marking	For 1-mark ques - 1/3 marks will be deducted; For 2 marks ques - 2/3 marks will be deducted.	No negative marking	No negative marking
Types of Question	Multiple Choice Questions (MCQ)	Multiple Select Questions (MSQ)	Numerical Answer Type (NAT)

In all sections, questions not attempted will result in zero mark. In Section-A (MCQ), wrong answer will result in **negative** marks. For each wrong answer to 1-mark questions, 1/3 mark will be deducted and similarly for each wrong answer to 2-mark questions, 2/3 mark will be deducted. In Section-B (MSQ), there are **no negative** and **no partial** marking provisions. There is **no negative** marking in Section-C (NAT) as well.

CUT OFF

	GENERAL	OBC-NCL/EWS	SC/ST/PwD	Organizing Institute
2025	8.75	7.87	4.37	IIT Delhi
2024	23.03	20.72	11.51	IIT Madras
2023	14.61	13.15	11.53	IIT Guwahati
2022	16.39	14.75	8.20	IIT Roorkee
2021	22.07	19.87	19.87	IISC Bangalore
2020	19.96	17.96	9.98	IIT Kanpur
2019	16.08	14.47	8.04	IIT Kharagpur
2018	18.97	17.07	9.48	IIT Bombay
2017	19.04	17.14	9.52	IIT Delhi

S Y L L A B U S

The Mathematical Statistics (MS) Test Paper comprises following topics of Mathematics (about 30% weight) and Statistics (about 70% weight).

MATHEMATICS:

Sequences and Series of real numbers: Sequences of real numbers, their convergence, and limits. Cauchy sequences and their convergence. Monotonic sequences and their limits. Limits of standard sequences. Infinite series and its convergence, and divergence. Convergence of series with non-negative terms. Tests for convergence and divergence of a series. Comparison test, limit comparison test, D'Alembert's ratio test, Cauchy's nth root test, Cauchy's condensation test and integral test. Absolute convergence of series. Leibnitz's test for the convergence of alternating series. Conditional convergence. Convergence of power series and radius of convergence.

Differential Calculus of one and two real variables: Limits of functions of one real variable. Continuity and differentiability of functions of one real variable. Properties of continuous and differentiable functions of one real variable. Rolle's theorem and Lagrange's mean value theorems. Higher order derivatives, Lebnitz's rule and its applications. Taylor's theorem with Lagrange's and Cauchy's form of remainders. Taylor's and Maclaurin's series of standard functions. Indeterminate forms and L' Hospital's rule. Maxima and minima of functions of one real variable, critical points, local maxima and minima, global maxima and minima, and point of inflection. Limits of functions of two real variables. Continuity and differentiability of functions of two real variables. Properties of continuous and differentiable functions of two real variables. Partial differentiation and total differentiation. Lebnitz's rule for successive differentiation. Maxima and minima of functions of two real variables. Critical points, Hessian matrix, and saddle points. Constrained optimization techniques (with Lagrange multiplier).

Integral Calculus: Fundamental theorems of integral calculus (single integral). Lebnitz's rule and its applications. Differentiation under integral sign. Improper integrals. Beta and Gamma integrals: properties and relationship between them. Double integrals. Change of order of integration. Transformation of variables. Applications of definite integrals. Arc lengths, areas and volumes.

Matrices and Determinants: Vector spaces with real field. Subspaces and sum of subspaces. Span of a set. Linear dependence and independence. Dimension and basis. Algebra of matrices. Standard matrices (Symmetric and Skew Symmetric matrices, Hermitian and Skew Hermitian matrices, Orthogonal and Unitary matrices, Idempotent and Nilpotent matrices). Definition, properties and applications of

determinants. Evaluation of determinants using transformations. Determinant of product of matrices. Singular and non-singular matrices and their properties. Trace of a matrix. Adjoint and inverse of a matrix and related properties. Rank of a matrix, row-rank, column-rank, standard theorems on ranks, rank of the sum and the product of two matrices. Row reduction and echelon forms. Partitioning of matrices and simple properties. Consistent and inconsistent system of linear equations. Properties of solutions of system of linear equations. Use of determinants in solution to the system of linear equations. Cramer's rule. Characteristic roots and Characteristic vectors. Properties of characteristic roots and vectors. Cayley Hamilton theorem.

STATISTICS:

Probability: Random Experiments. Sample Space and Algebra of Events (Event space). Relative frequency and Axiomatic definitions of probability. Properties of probability function. Addition theorem of probability function (inclusion exclusion principle). Geometric probability. Boole's and Bonferroni's inequalities. Conditional probability and Multiplication rule. Theorem of total probability and Bayes' theorem. Pairwise and mutual independence of events.

Univariate Distributions: Definition of random variables. Cumulative distribution function (c.d.f.) of a random variable. Discrete and Continuous random variables. Probability mass function (p.m.f.) and Probability density function (p.d.f.) of a random variable. Distribution (c.d.f., p.m.f., JAM 2024 MATHEMATICAL STATISTICS p.d.f.) of a function of a random variable using transformation of variable and Jacobian method. Mathematical expectation and moments. Mean, Median, Mode, Variance, Standard deviation, Coefficient of variation, Quantiles, Quartiles, Coefficient of Variation, and measures of Skewness and Kurtosis of a probability distribution. Moment generating function (m.g.f.), its properties and uniqueness. Markov and Chebyshev inequalities and their applications.

Standard Univariate Distributions: Degenerate, Bernoulli, Binomial, Negative binomial, Geometric, Poisson, Hypergeometric, Uniform, Exponential, Double exponential, Gamma, Beta (of first and second type), Normal and Cauchy distributions, their properties, interrelations, and limiting (approximation) cases.

Multivariate Distributions: Definition of random vectors. Joint and marginal c.d.f.s of a random vector. Discrete and continuous type random vectors. Joint and marginal p.m.f., joint and marginal p.d.f.. Conditional c.d.f., conditional p.m.f. and conditional p.d.f.. Independence of random variables. Distribution of functions of random vectors using transformation of variables and Jacobian method. Mathematical expectation of functions of random vectors. Joint moments, Covariance and Correlation. Joint moment

generating function and its properties. Uniqueness of joint m.g.f. and its applications. Conditional moments, conditional expectations and conditional variance. Additive properties of Binomial, Poisson, Negative Binomial, Gamma and Normal Distributions using their m.g.f.

Standard Multivariate Distributions: Multinomial distribution as a generalization of binomial distribution and its properties (moments, correlation, marginal distributions, additive property). Bivariate normal distribution, its marginal and conditional distributions and related properties.

Limit Theorems: Convergence in probability, convergence in distribution and their inter relations. Weak law of large numbers and Central Limit Theorem (i.i.d. case) and their applications.

Sampling Distributions: Definitions of random sample, parameter and statistic. Sampling distribution of a statistic. Order Statistics: Definition and distribution of the rth order statistic (d.f. and p.d.f. for i.i.d. case for continuous distributions). Distribution (c.d.f., p.m.f., p.d.f.) of smallest and largest order statistics (i.i.d. case for discrete as well as continuous distributions). Central Chi-square distribution: Definition and derivation of p.d.f. of central $\chi 2$ distribution with n degrees of freedom (d.f.) using m.g.f.. Properties of central $\chi 2$ distribution, additive property and limiting form of central $\chi 2$ distribution. Central Student's $\boldsymbol{t}$-distribution: Definition and derivation of p.d.f. of Central Student's t-distribution with n d.f., Properties and limiting form of central $t$distribution. Snedecor's Central $\boldsymbol{F}$-distribution: Definition and derivation of p.d.f. of Snedecor's Central F-distribution with (m, n) d.f.. Properties of Central F-distribution, distribution of the reciprocal of F-distribution. Relationship between t, F and $\chi 2$ distributions.

Estimation: Unbiasedness. Sufficiency of a statistic. Factorization theorem. Complete statistic. Consistency and relative efficiency of estimators. Uniformly Minimum variance unbiased estimator (UMVUE). Rao-Blackwell and Lehmann-Scheffe theorems and their applications. Cramer-Rao inequality and UMVUEs. Methods of Estimation: Method of moments, method of maximum likelihood, invariance of maximum likelihood estimators. Least squares estimation and its applications in simple linear regression models. Confidence intervals and confidence coefficient. Confidence intervals for the parameters of univariate normal, two independent normal, and exponential distributions.

Testing of Hypotheses: Null and alternative hypotheses (simple and composite), Type-I and Type-II errors. Critical region. Level of significance, size and power of a test, p-value. Most powerful critical regions and most powerful (MP) tests. Uniformly most powerful (UMP) tests. Neyman Pearson Lemma (without proof) and its applications to construction of MP and UMP tests for parameter of single parameter parametric families. Likelihood ratio tests for parameters of univariate normal distribution.

I N D E X

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NUMBER OF QUESTION ASKED IN IIT JAM - STATISTICS (2005 to 2025)

REAL ANALYSIS		2005	2006	2007	2008	2009	2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
1	Functions & Their Properties	-	-	-	-	-	-	1	-	-	1	5	-	-	1	1	-	2	-	-	1	2
2	Sequence of Real Numbers	-	-	1	-	-	-	-	1	2	-	2	1	2	5	3	4	7	3	2	2	2
3	Series of Real Numbers	-	-	-	1	1	-	1	1	-	2	1	3	4	2	2	1	2	1	2	3	1
4	Limit, Continuity, Differentiability & Application	-	-	1	2	1	2	2	-	2	2	4	6	6	5	5	5	3	3	2	4	3
5	Functions of Several Variables	-	-	-	-	-	-	-	1	1	1	1	-	-	1	1	2	2	3	3	2	1

CHAPTER-1 REAL ANALYSIS

Unit 1.1: Functions & Their Properties

Multiple Choice Questions (MCQ)

[2011: 6 marks]

1. The equation $x^{13} - e^{-x} + x - sinx = 0$ has
   (A) no real root.
   (B) more than two real roots.
   (C) exactly two real roots.
   (D) exactly one real root.

[2014: 1 Mark]

2. The equation
   $3x^2 - 12x + 11 + \frac{1}{5}(x^3 - 6x^2 + 11x - 6) = 0$
   has
   (A) exactly one root in the interval (1, 2)
   (B) exactly two distinct roots in the interval (1, 2)
   (C) exactly three distinct roots in the interval (1, 2)
   (D) NO roots in the interval (1, 2)

[2015: 1 Mark]

3. Which of the following statements is true for all real numbers x?
   (A) $e^{-x} \leq 1 - x$ (B) $e^{-x} \geq 1 - x$
   (C) $e^{x} \geq 1 - x$ (D) $e^{x} \leq 1 - x$

[2015: 2 Marks]

4. Let $f:(0, \infty) \to \mathbb{R}$ be given by $f(x) = \log x - x + 2$. Then, the number of roots of f is
   (A) 0 (B) 1
   (C) 2 (D) 3

[2018: 2 Marks]

5. Let $f, g: \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2 - \frac{\cos x}{2}$ and $g(x) = \frac{x \sin x}{2}$, then
   (A) f(x) = g(x) for more than two values of x
   (B) f(x) ≠ g(x), for all x in $\mathbb{R}$
   (C) f(x) = g(x) for exactly one value of x
   (D) f(x) = g(x) for exactly two values of x

[2019: 2 Marks]

6. A possible value of b∈ $\mathbb{R}$ for which the equation $x^4 + bx^3 + 1 = 0$ has no real root is
   (A) $\frac{-11}{5}$ (B) $\frac{-3}{2}$
   (C) 2 (D) $\frac{5}{2}$

[2021: 1 Mark]

7. Let $f: \mathbb{R} \to \mathbb{R}$ be a function defined by
   $$f(x) = x^7 + 5x^3 + 11x + 15, x \in \mathbb{R}.$$
   Then, which of the following statements is TRUE?
   (A) f is both one-one and onto
   (B) f is neither one-one nor onto
   (C) f is one-one but NOT onto
   (D) f is onto but NOT one-one

[2024: 2 Marks]

8. Let $\Omega = \{1,2,3,4,5,6\}$. Then which of the following classes of sets is algebra?
   (A) $\mathcal{F}_1 = \{\emptyset, \Omega, \{1,2\}, \{3,4\}, \{3,6\}\}$
   (B) $\mathcal{F}_2 = \{\emptyset, \Omega, \{1,2,3\}, \{4,5,6\}\}$
   (C) $\mathcal{F}_3 =$
   $\{\emptyset, \Omega, \{1,2\}, \{4,5\}, \{1,2,4,5\}, \{3,4,5,6\}, \{1,2,3,6\}\}$
   (D) $\mathcal{F}_4 = \{\emptyset, \{4,5\}, \{1,2,3,6\}\}$

[2025: 1 Mark]

9. Let $f: \mathbb{R} \to \mathbb{R}$ be a continuous odd function that is not identically zero. Further, suppose that f is a periodic function. Define
   $$g(x) = \int_0^x f(t)\, dt.$$
   Then
   (A) g is odd and not periodic.
   (B) g is odd and periodic.
   (C) g is even and not periodic.
   (D) g is even and periodic.

[2025: 2 Marks]

10. Let $f: \mathbb{R} \to \mathbb{R}$ be given by $f(x) = x + \pi\cos x$. Then the number of solutions of the equation $f(x) = 0$ is
    (A) 1 (B) 2
    (C) 3 (D) 4

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D	A	B	C	D	B	A	B	D	C

EXPLANATION

Multiple Choice Questions (MCQ)

1. Explanation: From Bolzano's Intermediate Value Theorem, there exists a point c ∈ (0,1) such that f(c) = 0

i.e., c is a real root of the given equation.

$f(x) = x^{13} - e^{-x} + x - sinx$

$f'(x) = 13x^{12} + e^{-x} + 1 - cosx > 0$, for all $x \in \mathbb{R}$,

which implies that $f(x)$ is strictly increasing on $\mathbb{R}$, and therefore, $f(x) < f(c) = 0, \forall\, x \epsilon\, c$ and $f(x) > f(c) = 0, \forall\, x \epsilon\, c$.

Therefore, the given equation has exactly one real root at $x = c$.

⇒ option (D) is correct

2. Explanation:

$3x^2 - 12x + 11 + \frac{1}{5}(x^3 - 6x^2 + 11x - 6) = 0$

$\Rightarrow f(x) = x^3 + 9x^2 - 49x + 49 = 0$

$f'(x) = 3x^2 + 18x - 49 = 0$

$\Rightarrow x = \frac{-18 \pm \sqrt{18^2 + 4.3.49}}{2.3}$

$x = 2.0033, -8.0033$

$f(1.1) = 7.3 > 0$

$f(1.9) = -4.75 < 0$

∴ The given equation has one root in the interval of $(1,2)$

⇒ Option (A) is correct.

3. Explanation:

$f(x) = e^{-x} - (1 - x),\ x \in \mathbb{R}$

$f'(x) = -e^{-x} + 1,\ x \in \mathbb{R}$

$f'(x) \leqslant 0$ for $x \leqslant 0$

$f'(x) \geqslant 0$ for $x \geqslant 0$

& $f'(x) \geqslant 0$ for $x \geqslant 0$

Hence $f(x)$ is decreasing on $(-\infty, 0]$ is increasing on $[0, \infty)$

Thus, $f(x)$ has minima at $x = 0$ &

Therefore, $f(x) \geqslant f(0)$ for all $x \in \mathbb{R}$.

$\Longrightarrow e^{-x} \geqslant 1 - x$ for all $x \in \mathbb{R}$.

⇒ option (B) is correct.

4. Explanation:

$f: (0, \infty) \to \mathbb{R}$ be given by

$f(x) = \log\, x - x + 2$

$f'(x) = \frac{1}{x} - 1\ x \in (0, \infty)$.

∴ $f(x) > 0$ for $x \in (0,1)$ &

$f'(x) < 0$ for $x \in (1, \infty)$

Hence $f(x)$ is strictly increasing in $x \in (0,1)$ & $f'(x)$ is strictly decreasing in $(1, \infty)$.

$$\lim_{x \to 0} f(x) = \lim_{x \to 0} (\log x - x + 2) \ \ \&$$

$$f(1) = \log\, 1 - 1 + 2 \ = 1 > 0$$

∴ the number of roots of f is 2.

⇒ option (C) is correct.

5. Explanation:

$$f(x) = \frac{x^2 - \cos\, x}{2} \text{ and } g(x) = \frac{x \sin\, x}{2}$$

Put $F(x) = \mathrm{f} - \mathrm{g} = \frac{x^2 - \cos\, x}{2} - \frac{x \sin\, x}{2}$

Now put $x = 0, \pm\pi\ , \pm\frac{\pi}{2}$,

By Intermediate theorem we get,

$$F(0) = \frac{-1}{2} < 0$$

$$F\left(\frac{\pi}{2}\right) > 0\ x \in (0, \infty)$$

∴ It has exactly one real root.

∴ $F(x) = 0$ (by intermediate theorem)

Now put $x = -\frac{\pi}{2}$.

$\Rightarrow\ F\left(-\frac{\pi}{2}\right) > 0, \quad x \in \left(-\frac{\pi}{2}, 0\right)$

$\Rightarrow\ F(x) = 0.$

∴ It has exactly one real root.

How, Diff f w.r.t. x.

$$F'(x) = 2x + \frac{\sin\, x}{2} - \frac{\sin\, x}{2} - \frac{x \cos\, x}{2}.$$

$$F'(x) = 2x - \frac{x \cos\, x}{2}$$

$$F'(x) = x\left(2 - \frac{\cos\, x}{2}\right)$$

If $x \in (-\infty, 0)$ then F is decreasing and $x \in (0, \infty)$ then F is monotonically increasing.

It has exactly two roots.

Hence option D is correct.

6. Explanation:

Observation: $x^4 + bx^3 + 1 = 0$

Let's assume ' b ' is positive in that case for x is negative number.

The given equation can be zero for some value.

⇒ for no real roots if b is negative. now, taking $b = -3/2$

$\Rightarrow x^4 - \frac{3}{2}x^3 + 1 = 0$

$\Rightarrow 2x^4 - 3x^3 + 2 = 0$

Note → for all $x < 0$, 0 Given equation is positive.

also, for $x = 0$

observation at $x = 1,2,3$ only

1 = positive
2 = positive
3 = positive .

i.e. equation is always positive

$\therefore b = -3/2$ has no real root

∴ option B is correct.

7. Explanation:

$$f(x) = x^7 + 5x^3 + 11x + 15, x \in \mathbb{R}$$

$$f'(x) = 7x^6 + 15x^2 + 11x$$

$\Rightarrow f'(x) > 0 \; \forall f \; x \in \mathbb{R}$

$\therefore f(x)$ is increasing function.

∴ f(x) is one-one function.

Now, for onto:

$x \in \mathbb{R} \Rightarrow f(x) \in \mathbb{R}$

∴ Range → codomain

∴ mapping is onto

∴ f is both one-one and onto

Hence option A is correct.

8. Explanation:

Given $\Omega = \{1,2,3,4,5,6\}$

F is Field (Algebra) iff.

(a) $\phi \in F$

(b) If $A \in F \Rightarrow A^C \in F$

(c) $A_1, A_2, \ldots, A_n \in F$

$\prod \cup_{k=1}^{n} A_k \in f$

∴ The classes of sets is algebra

$\sigma \{\phi, \Omega, \{1,2,3\}, \{4,5,6\}\}$

⇒ Option (B) is correct.

9. Explanation:

We are given:

- $f: \mathbb{R} \to \mathbb{R}$ is continuous, odd, periodic, not identically zero
- Define: $g(x) = \int_0^x f(t)\, dt$

Let's compute: $g(-x) = \int_0^{-x} f(t)dt = -\int_{-x}^{0} f(t)dt$

Now change variable $t = -u \Rightarrow dt = -du$

Then:

$$\int_{-x}^{0} f(t)\, dt = \int_{x}^{0} f(-u)(-du) = \int_{0}^{x} f(-u)du$$

Since f is odd, $f(-u) = -f(u)$, so:

$$g(-x) = -\int_0^x f(-u)\, du = -\int_0^x (-f(u))\, du$$

$$= \int_0^x f(u)\, du = g(x)$$

So: $g(-x) = g(x) \Rightarrow g$ is **even**

Since f is periodic with period T, and:

$g(x+T) = \int_0^{x+T} f(t)dt = \int_0^T f(t)\, dt +$

$\int_T^{x+T} f(t)\, dt$

Using periodicity of f: $\int_T^{x+T} f(t)dt = \int_0^x f(t)dt$

And since f is odd, $\int_0^T f(t)\, dt = 0$

So: $g(x+T) = 0 + \int_0^x f(t)dt = g(x)$

Hence, g is periodic

Correct answer: (D) g is even and periodic

10. Explanation: (C)

Given: $f(x) = x + \pi\cos\, x = 0$

Rearranged as : $x = -\pi\cos\, x \Rightarrow |x| \leq \pi$

(Since $|\cos\, x| \leq 1$)

Check values:

- $f(-\pi) = -2\pi < 0$,
- $f(0) = \pi > 0$

  ⇒ 1 root in (-π, 0)
- $f(\pi) = 0 \Rightarrow$ root at π
- $f(x) > 0$ in (0, π), so no root there

Function is odd, so root at $x = -r$ if root at $x = r$

So we get:

- 1 root in (-π, 0)
- 1 root at 0 < x < π
- 1 root at x = π

Answer: (C) 3 real solutions

Multiple Select Questions (MSQ)

[2015: 2 Marks]

1. Let $f: \mathbb{R} \to \mathbb{R}$ Define g: $\mathbb{R} \to \mathbb{R}$ by

g(x)= f(x) (f(x)+f(−x)).

Then

(A) g is even for all f (B) g is odd for all f

(C) g is even if f is even (D) g is even if f is odd

Answer Key
1
C, D

EXPLANATION

Multiple Select Questions (MSQ)

1. Explanation:

$f: \mathbb{R} \to \mathbb{R}$. Define $g: \mathbb{R} \to \mathbb{R}$.

$$g(x) = f(x)(f(x) + f(-x))$$

Putting $x = -x$ then we have.

$$g(-x) = f(-x)(f(-x) + f(x)).$$

Let f is even i.e. $f(-x) = f(x)$.

$$g(x) = 2(f(x))^2 = g(-x).$$

Hence g is even.

Let f is odd. i.e. $f(-x) = -f(x)$.

$g(x) = 0 = g(-x)$.

Hence g is even.

⇒ options (C) &(D) are correct.

Numerical Answer Type (NAT)

[2015: 1 Mark]

1. Let [x] be the greatest integer less than or equal to x. Then $\int_{-1}^{2} [x]$dx_____

[2015: 1 Mark]

2. The number of real solutions of the equation $x^3 + 3x^2 + 3x + 7 = 0$ is _____

[2021: 2 Marks]

3. The number of real roots of the polynomial f(x)= x^{11}- 13 x +5 is _____

Answer Key		
1	**2**	**3**
0	1	3

EXPLANATION

Numerical Answer Type (NAT)

1. Explanation

Using the definition of $[x]$, we have.

$$\int_{-1}^{2} [x]dx = \int_{-1}^{0} (-1)dx + \int_{0}^{1} 0dx + \int_{1}^{2} 1dx$$

$= [-x]_{-1}^{0} + 0 + [x]_{1}^{2}$

$= -1 + 0 + 1$

$= 0$.

2. Explanation

$$f(x) = x^3 + 3x^2 + 3x + 7$$

$$f'(x) = 3x^2 + 6x + 3$$

$= 3(x+1)^2 > 0$ for all $x \in \mathbb{R}\{-1\}$

We have, $f(-3) = -2 <$

0 & $f(1) = 14 > 0$

then, there exists a $c \in (3,1)$ such that $f(c) = 0$

Hence, $x^3 + 3x^2 + 3x + 7$ has only one real solution

3. Explanation:

$$f(x) = x'' - 13x + 5$$

By opposite sign theorem

$$\left.\begin{matrix} f(0) = 5 > 0 \\ f(1) = 1 - 13 + 5 < 0 \end{matrix}\right] \to 1 \text{ root}$$

$$\left.\begin{matrix} F(1) = 1 - 13 + 5 < 0 \\ F(2) = 2^4 - 26 + 570 \end{matrix}\right] \to \ 1root$$

Now

$$\left.\begin{matrix} \mathrm{F}(-1) = -1 + 13 + 5 > 0 \\ f(-2) = -2^4 + 26 + 5 < 0 \end{matrix}\right] \to 1 \text{ root}$$

∴ Hence, we have total 3 roots which are real root

Unit 1.2: Sequence of Real Numbers

Multiple Choice Questions (MCQ)

[2007: 6 marks]

1. $\lim_{n\to\infty} \frac{1-2+3-4+5-6+\cdots..+(-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$ equals

(A) ∞ (B) ½

(C) 0 (D)-1/2

[2012: 6 marks]

2. Let L = $\lim_{n\to\infty} n\ [f\left(\frac{1}{n}\right)+ f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{k}{n}\right)-k\,f(0)]$, where k is positive integer. if f (x)= sin x, then L is equal to

(A) $\frac{(k+1)(k+2)}{6}$ (B) $\frac{(k+1)(k+2)}{2}$

(C) $\frac{k(k+1)}{2}$ (D) k (k + 1)

[2013: 2 Marks]

3. Let $\{a_n\}_{n\geq 1}$ be a sequence of positive real numbers such that $lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{2}$. Then $lim_{n\to\infty}\frac{e^{a_n^2}+a_n}{4}$ is equal to

(A) ∞ (B) $\frac{e^{\frac{1}{4}}}{4}+\frac{1}{8}$

(C) $\frac{e^{\frac{1}{4}}}{4}$ (D) $\frac{1}{4}$

[2015: 1 Mark]

4. If a sequence {X_n} is monotone and bounded, then

(A) there exists a subsequence of {X_n} that diverges

(B) there may exist a subsequence of {X_n} that is not monotone

(C) all subsequences of {X_n} converge to the same limit

(D) there exist at least two subsequences of {X_n} which converge to distinct limits

[2015: 1 Mark]

5. Let f: ℝ → ℝ be defined by f(x) = x(x-1)(x-2). Then

(A) f is one-one and onto

(B) f is neither one-one nor onto

(C) f is one-one but not onto

(D) f is not one-one but onto

[2017: 1 Mark]

6. Let $\{a_n\}_{n\geq 1}$ be a sequence defined as follows:

$a_1 = 1$ and $a_{n+1}=\frac{7a_n+11}{21}$, n ∈ N.

Which of the following statements is TRUE?

(A) $\{u_n\}_{n\geq 1}$ is an increasing sequence which diverges

(B) $\{u_n\}_{n\geq 1}$ is an increasing sequence with $=\frac{11}{14}$

(C) $\{u_n\}_{n\geq 1}$ is a decreasing sequence which diverges

(D) $\{u_n\}_{n\geq 1}$ is a decreasing sequence with $a_n=\frac{11}{14}$

[2018: 1 Mark]

7. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 2$ and, for n ≥ 1, $a_{n+1} = \frac{2a_n+1}{a_n+1}$

Then

(A) 1.5 ≤ an ≤ 2, for all natural number n ≥ 1

(B) there exists a natural number n ≥ 1 such that an > 2

(C) there exists a natural number n ≥ 1 such that an < 1.5

(D) there exists a natural number n ≥ 1 such that $a_n = \frac{1+\sqrt{5}}{2}$

[2018: 1 Mark]

8. $(1+\frac{2}{n})^{n^2}\ e^{-2n}$ is

(A) e^{-2} (B) e^{-1}

(C) e (D) e^2

[2018: 1 Mark]

9. Let $\{a_n\}_{n\geq 1}$ and Let $\{b_n\}_{n\geq 1}$ be two convergent sequences of real numbers. For n ≥ 1, define u_n = max{a_n, b_n} and v_n= min{a_n, b_n}. Then

(A) Neither $\{u_n\}_{n\geq 1}$ nor $\{v_n\}_{n\geq 1}$ converges

(B) $\{u_n\}_{n\geq 1}$ converges but $\{v_n\}_{n\geq 1}$ does not converge

(C) $\{u_n\}_{n\geq 1}$ does not converge but $\{v_n\}_{n\geq 1}$converges

(D) both $\{u_n\}_{n\geq 1}$ and $\{v_n\}_{n\geq 1}$ $converge$.

[2019: 2 Marks]

10. $\lim_{n\to\infty} \frac{1+\frac{1}{2}+\cdots+\frac{1}{n}}{(\pi^n+e^n)^{1/n}\log_e n}$ $equals$

(A) $\frac{1}{\pi}$ (B) $\frac{1}{e}$

(C) $\frac{e}{\pi}$ (D) $\frac{\pi}{e}$

[2020: 1 Mark]

11. If $\{x_n\}_{n\geq 1}$ is a sequence of real numbers such that $lim_{n\to\infty}\frac{x_n}{n}=0.001$, then

(A) $\{x_n\}_{n\geq 1}$ is a bounded sequence

(B) $\{x_n\}_{n\geq 1}$ is an unbounded sequence

(C) $\{x_n\}_{n\geq 1}$ is a convergent sequence

(D) $\{x_n\}_{n\geq 1}$ is a monotonically decreasing sequence

[2020: 2 Marks]

12. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_1 = 1, a_2 = 7$ and $a_{n+1}=\frac{a_n+a_{n-1}}{2}, n\geq 2$. Assuming that $lim_{n\to\infty}a_n$ exists, the value of $lim_{n\to\infty}a_n$ is

(A) $\frac{19}{4}$ (B) $\frac{9}{2}$

(C) 5 (D) $\frac{21}{4}$

[2021: 1 Mark]

13. The value of the limit

$$\lim_{n\to\infty}\left(\left(1+\frac{1}{n}\right)\left(1+\frac{2}{n}\right)\cdots\left(1+\frac{n}{n}\right)\right)^{\frac{1}{n}}$$

is equal to

(A) e (B) $\frac{1}{e}$

(C) $\frac{3}{e}$ (D) $\frac{4}{e}$

[2021: 1 Mark]

14. The value of the limit

$$\lim_{n\to\infty}\sum_{k=0}^{n}\binom{2n}{k}\frac{1}{4^n}$$

is equal to

(A) 1 (B) $\frac{1}{2}$

(C) 0 (D) $\frac{1}{4}$

[2021: 2 Marks]

15. Let a_1=5 and define recursively

$$a_{n+1}=3^{\frac{1}{4}}(a_n)^{\frac{3}{4}},\ n\geq 1.$$

Then,which of the following statements is **TRUE?**

(A) $\{a_n\}$ is monotone increasing and $\lim_{n\to\infty} a_n$ =3

(B) $\{a_n\}$ is monotone decreasing and $\lim_{n\to\infty} a_n$ =3

(C) $\{a_n\}$ is non-monotone and $\lim_{n\to\infty} a_n$ =3

(D) $\{a_n\}$ is decreasing and $\lim_{n\to\infty} a_n$ = 0

[2021: 2 Marks]

16. Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_n \geq 1$ for all n ≥ 1. Then, which of the following conditions imply the divergence of $\{a_n\}_{n\geq 1}$?

(A) $\{a_n\}_{n\geq 1}$ is non-increasing

(B) $\sum_{n=1}^{\infty} b_n$ converges, where $b_1 = a_1$ and $b_n = a_{n+1} - a_n$, for all $n > 1$

(C) $\lim_{n\to\infty}\frac{a_{2n+1}}{a_{2n}} = \frac{1}{2}$

(D) $\left\{\sqrt{a_n}\right\}_{n\geq 1}$ converges

[2022: 1 Mark]

17. Let $\{a_n\}_{n\geq 1}$ be a sequence of non-zero real numbers. Then which one of the following statements is true?

(A) If $\left\{\frac{a_{n+1}}{a_n}\right\}_{n\geq 1}$ is a convergent sequence, then $\{a_n\}_{n\geq 1}$ is also a convergent sequence.

(B) If $\{a_n\}_{n\geq 1}$ is a bounded sequence, then $\{a_n\}_{n\geq 1}$ is a convergent sequence.

(C) If $|a_{n+2}-a_{n+1}| \leq \frac{3}{4}|a_{n+1} - a_n|$ for all n ≥ 1, then $\{a_n\}_{n\geq 1}$ is a Cauchy sequence.

(D) If $\{|an|\}_{n\geq 1}$ is a Cauchy sequence, then $\{a_n\}_{n\geq 1}$ is also a Cauchy sequence.

[2022: 2 Marks]

18. $\lim_{n\to\infty}\frac{6}{n+2}\left\{\left(2+\frac{1}{n}\right)^2+\left(2+\frac{2}{n}\right)^2+\cdots+\left(2+\frac{n-1}{n}\right)^2\right\}$ equals

(A) 38 (B) 36

(C) 32 (D) 30

[2023: 2 Marks]

19. Let $\{a_n\}_{n\geq 1}$be a sequence such that a_1=1 and $4a_{n+1}=\sqrt{45+16a_n}$,n=1,2,3,…….

Then, which one of the following statements is TRUE?

(A) $\{a_n\}_{n\geq 1}$ is monotonically increasing and converges to to $\frac{17}{8}$

(B) $\{a_n\}_{n\geq 1}$ is monotonically increasing and converges to to $\frac{9}{4}$

(C) $\{a_n\}_{n\geq 1}$ is bounded above by $\frac{17}{8}$

(D) $\sum_{n=1}^{\infty} a_n$ is convergent

[2024: 1 Mark]

20. Let $a_n = 1+\frac{1}{2}+\cdots+\frac{1}{n}$ and $b_n = \frac{n^2}{2^n}$ for all $n \in \mathbb{N}$. Then

(A) $\{a_n\}$ is a Cauchy sequence but $\{b_n\}$ is NOT a Cauchy sequence

(B) $\{a_n\}$ is NOT a Cauchy sequence but $\{b_n\}$ is a Cauchy sequence

(C) both $\{a_n\}$ and $\{b_n\}$ are Cauchy sequences

(D) neither $\{a_n\}$ nor $\{b_n\}$ is a Cauchy sequence

[2025: 1 Mark]

21. Let $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ be sequences given by

$$a_n = \left\lfloor\frac{n^2}{n+1}\right\rfloor \text{ and } b_n = \frac{n^2}{n+1} - a_n.$$

Then

(A) $\{a_n\}_{n\geq 1}$ converges and $\{b_n\}_{n\geq 1}$ diverges.

(B) $\{a_n\}_{n\geq 1}$ diverges and $\{b_n\}_{n\geq 1}$ converges.

(C) Both $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ diverge.

(D) Both $\{a_n\}_{n\geq 1}$ and $\{b_n\}_{n\geq 1}$ converge.

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D	C	D	C	D	D	A	A	D	A
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
B	C	D	B	B	C	C	A	B	B
21									
B									

EXPLANATIONS

Multiple Choice Questions (MCQ)

1. Explanation:

$$\lim_{n\to\infty}\frac{1-2+3-4+\cdots+(-2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{(1+3+5+\cdots+2n-1)-(2+4+6+\cdots+2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{(1+2+3+4+\cdots+2n-1+2n)-2(2+4+6+\cdots+2n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{\frac{2n(2n+1)}{2}-4(1+2+3+4+\cdots+n)}{\sqrt{n^2+1}+\sqrt{n^2-1}}$$

$$=\lim_{n\to\infty}\frac{n(2n+1)-4\frac{n(n+1)}{2}}{n\sqrt{1+\frac{1}{n^2}}+n\sqrt{1-\frac{1}{n^2}}}$$

$$=\lim_{n\to\infty}\frac{n\{(2n+1)-2(n+1)\}}{n\left\{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}\right\}}$$

$$=\lim_{n\to\infty}\frac{2n+1-2(n+1)}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}}$$

$$=\lim_{n\to\infty}\frac{-1}{\sqrt{1+\frac{1}{n^2}}+\sqrt{1-\frac{1}{n^2}}}=\frac{-1}{\sqrt{1}+\sqrt{1}}=\frac{-1}{2}$$

⇒ option (D) is correct.

2. Explanation:

$L=\lim_{n\to\infty}n\left[f\left(\frac{1}{n}\right)+f\left(\frac{2}{n}\right)+\cdots+f\left(\frac{k}{n}\right)-kf(0)\right]$

$f(x)=\sin x$

$Take\ \ n=\frac{1}{m}\Rightarrow m=\frac{1}{n}.$

$L=\lim_{m\to0}\frac{1}{m}\left[f(m)+f(2m)+\cdots+f(km)-kf(0)\right]$

$=\lim_{m\to0}\frac{1}{m}\left[\sin(mx)+\sin 2mx+\cdots+\sin km\right\}-k\sin(0)]$

$=\lim_{m\to0}\left[\frac{\sin mx}{m}+\frac{2\sin 2mx}{2m}+\cdots+\frac{\sin km}{km}\right]$

$=1+2+\cdots+k-0$

$L=\frac{k(k+1)}{2}$

∴ Option (c) is correct.

3. Explanation:

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{1}{2}$

We Take

$a_n=\frac{1}{2^n}\quad a_{n+1}=\frac{1}{2^{n+1}}$

$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\frac{\frac{1}{2^{n+1}}}{\frac{1}{2^n}}=\frac{1}{2}$

$\lim_{n\to\infty}\frac{e^{a_{n^2}}+a_n}{4}=\lim_{n\to\infty}\frac{e^{\frac{1}{2^{2n}}}+\frac{1}{2^n}}{4}$

$\therefore\lim_{n\to\infty}\frac{1}{2^{2n}}=0\lim_{n\to\infty}\frac{1}{2^n}=0$

$\therefore\lim_{n\to\infty}\frac{\frac{1}{e^{2n}}+\frac{1}{2^n}}{4}=\frac{e^0+0}{4}$

$=\frac{1}{4}$

⇒ Option (D) is correct

4. Explanation:

Given, $\{x_n\}$ is montane & bounded sequences

if $\{x_n\}$ has converges to limit l. then all Subsequences of $\{x_n\}$ same limit of l.

⇒ option (c) is correct.

5. Explanation:

$f:\mathbb{R}\to\mathbb{R}$ be defined

$f(x)=x(x-1)(x-2).$

$f(x)$ is zero for $x=1,2$

Hence $f(x)$ is not one-one

also $\lim_{x\to-\infty}f(x)=-\infty,\lim_{x\to\infty}f(x)=\infty$

& $f(x)$ is contineous on $\mathbb{R}$.

Then From intermediate value Theorem such that $f(x)$ take all values in $(-\infty,\infty)$

$\therefore f(x)$ is onto.

⇒ f is not one-one but onto

⇒ option (D) is correct

6. Explanation:

$a_1=1\ a_{n+1}=\frac{7a_n+11}{21}$

Put $n=1$

$\Rightarrow a_2=\frac{7a_1+11}{21}=\frac{7+11}{21}=\frac{18}{21}<1$

Then we have

$a_{k+2}=\frac{7a_{k+2}+11}{21}<\frac{7a_k+11}{21}=a_{k+1}$

Hence, from mathematical induction,

We have

$a_{n+1}<a_n$

i.e. $\{a_n\}_{n\geq1}$ is decreasing sequences.

CHAPTER-2 LINEAR ALGEBRA

Unit 2.1: Matrices & Their Properties

Multiple Choice Questions (MCQ)

[2007: 6 Marks]

1. Let A be a 4×4 nonsingular matrix and B be the matrix obtained from A by adding to its third row twice the first row. Then $\det(2A^{-1}B)$ equals

(A) 2 (B) 4

(C) 8 (D) 16

[2009: 6 Marks]

2. Let $A = \langle(a_{ij})\rangle$ be an orthogonal matrix of order n such that $a_{1j} = \frac{1}{\sqrt{n}}, j = 1, \ldots, n$. If $\bar{a} = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}$, then $\sum_{i=1}^{n}\sum_{j=1}^{n}(a_{ij} - \bar{a})^2$ equals

(A) $\frac{n+1}{n}$ (B) $\frac{n-1}{n}$

(C) $\frac{n^2+1}{n}$ (D) $\frac{n^2-1}{n}$

[2010: 6 Marks]

3. Let P= $\frac{\underline{x}\,\underline{x}^T}{\underline{x}^T\underline{x}}$ be an n x n (n > 1) matrix, where x is a nonzero column vector. Then which one of the following statements is FALSE?

(A) P is idempotent (B) P is orthogonal

(C) P is symmetric (D) Rank of P is one

[2010: 6 Marks]

4. Let P = (p_{ij}) be a 50 x 50 matrix, where p_{ij} = min(i, j); i ,j = 1,..., 50. Then the rank of P equals

(A) 1 (B) 2

(C) 25 (D) 50

[2014: 1 Mark]

5. The matrix

M = $\begin{pmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{pmatrix}$

(A) is an elementary matrix

(B) can be written as a product of elementary matrices

(C) does NOT have linearly independent eigenvectors

(D) is a nilpotent matrix

[2015: 2 Marks]

6. Let M be the matrix $\begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$. Which of the following matrix equations does M satisfy?

(A) $M^2 + 3M + 5I=0$ (B) $M^3 - M^2 - 5M=0$

(C) $M^3 - 3M^2 + M=0$ (D) $M^2 - M + 5I =0$

[2015: 2 Marks]

7. If the determinant of an n x n matrix is zero, then

(A) rank(4) ≤ n-2

(B) the trace of 4 is zero

(C) zero is an eigenvalue of A

(D) x=0 is the only solution of Ax=0

[2015: 2 Marks]

8. The number of distinct real values of x for which the matrix

$$\begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$$

is singular is

(A) 1 (B) 2

(C) 3 (D) infinite

[2016: 1 Mark]

9. Let P = $\begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \\ 1 & 2 & -2 & -4 \end{bmatrix}$

Then rank of P equals

(A) 4 (B) 3

(C) 2 (D) 1

[2016: 1 Mark]

10. Let α, β, γ be real numbers such that $\beta \neq 0$ and $\gamma \neq 0$. Suppose $\begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$ and P^{-1} = P. Then

(A) $\alpha = 0$ and $\beta\gamma = 1$ (B) $\alpha \neq 0$ and $\beta\gamma = 1$

(C) $\alpha = 0$ and $\beta\gamma = 2$ (D) $\alpha = 0$and $\beta\gamma = -1$

[2016: 2 Marks]

11. Let P be a 3×3 singular matrix such that p$\vec{v} = \vec{v}$ for a nonzero vector $\vec{v}$ and

$$P\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ -2/5 \end{bmatrix}$$

Then

(A) $P^3 = \frac{1}{5} = (7P^2 - 2P)$ (B) $P^3 = \frac{1}{4}$ $(7\ P^2-2P)$

(C) $P^3 = \frac{1}{3}(7P^2-2P)$ (D) $P^3 = \frac{1}{2}(7P^2-2P)$

[2018: 1 Mark]

12. Let M = $\begin{bmatrix} \frac{1}{4} & \frac{3}{4} \\ \frac{3}{5} & \frac{2}{5} \end{bmatrix}$ If I is the 2 x 2 identity matrix and 0 is the 2 x 2 zero matrix, then

(A) $20\,M^2 - 13M + 7\,I = 0$ (B) $20\,M^2 - 13M - 7\,I = 0$

(C) $20\,M^2 + 13M + 7\,I = 0$ (D) $20\,M^2 + 13M + 7\,I = 0$

[2018: 2 Marks]

13. Let M = [X Y Z] be an orthogonal matrix with X, Y, Z $\in \mathbb{R}^3$ as its column vectors. Then Q= $XX^T + YY^T$

(A) is a skew-symmetric matrix

(B) is the 3 x 3 identity matrix

(C) satisfies $Q^2 = Q$

(D) satisfies QZ = Z

[2019: 2 Marks]

14. If p=$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 1 \\ 2 & 0 & -1 \end{bmatrix}$ and $6P^{-1} = aI_3 + bp - p^2$ then the ordered pair (a, b) is

(A) (3,2) (B) (2,3)

(C) (4,5) (D) (5,4)

[2020: 2 Marks]

15. Let M be an $n \times n$ non-zero skew symmetric matrix. Then the matrix $(I_n - M)(I_n + M)^{-1}$ is always

(A) singular (B) symmetric

(C) orthogonal (D) idempotent

[2022: 2 Marks]

16. Let $A = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix}$. Then the sum of all the elements of A^{100} equals

(A) 101 (B) 103

(C) 102 (D) 100

[2023: 1 Mark]

17. Let $M = M_1M_2$, where M_1 and M_2 are two 3 x 3 distinct matrices. Consider the following two statements:

(I) The rows of M are linear combinations of rows of M_2

(II) The columns of M are linear combinations of columns of M_1

Then,

(A) only (I) is TRUE

(B) only (II) is TRUE

(C) both (I) and (II) are TRUE

(D) neither (I) nor (II) is TRUE

[2024: 1 Mark]

18. Let $A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix}$ be a real matrix, where $ad = 1$ and $c \neq 0$. If $A^{-1} + (\text{adj } A)^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$ then $(\alpha, \beta, \gamma, \delta)$ is equal to

(A) $(a + d,\ 0,\ 0,\ a + d)$ (B) $(a + d,\ 0,\ c,\ a + d)$

(C) $(a, 0, 0, d)$ (D) $(a, 0, c, d)$

[2025: 2 Marks]

19. Let A be an $n \times n$ matrix. Which of the following statements is NOT necessarily true?

(A) If $\text{rank}(A^3) = \text{rank}(A^4)$, then $\text{rank}(A^6) = \text{rank}(A^7)$.

(B) If $\text{rank}(A) = n$, then it is possible to obtain a singular matrix by suitably changing a single entry of A.

(C) If $\text{rank}(A) = n$, then $\text{rank}(A + A^T) \geq \frac{n}{2}$.

(D) If $\text{rank}(A) < n$, then it is possible to obtain a nonsingular matrix by suitably changing $n - \text{rank}(A)$ entries of A.

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
D	A	B	D	B	B	C	B	C	A
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	
A	B	C	A	C	B	C	A	C	

EXPLANATIONS

Multiple Choice Questions (MCQ)

1. Explanation: Adding scalar multiple of one row to other does not change the determinant.

$\therefore \det A = \det B$

Now,

$\det(2A^{-1}B)$

$= 2^4 \det(A^{-1}B)$ $(\because \det(kM_{n\times n}) = k^n \det(A))$

$= 2^4 \det(A^{-1}) \det(B)$ $(\because \det(MN) = \det(M)\det(N))$

$= 2^4 \frac{1}{\det A} \det B$

$= 16$

$\Rightarrow$ option (D) is correct.

2. Explanation:

$$a_{ij} = \frac{1}{\sqrt{n}}, j = 1,2, \ldots, n.$$

$$\bar{a} = \frac{1}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} a_{ij}$$

$$n = 2$$

$$\bar{a} = \frac{1}{2^2}\sum_{i=1}^{2}\sum_{j=1}^{2} a_{ij}$$

$$aij = \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2^2}\sum_{i=1}^{2}\sum_{i=1}^{2} a_{ij}$$

$$= \frac{1}{2^2}\sum_{i=1}^{2}(a_{i1} + a_{i2})$$

$$= \frac{1}{2^2}(a_{11} + a_{12} + a_{21} + a_{22})$$

$$= \frac{1}{2^2}\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)$$

$$= \frac{1}{2\sqrt{2}}$$

$$\sum_{i=1}^{2}\sum_{i=1}^{2}\left[(a_{ij})^2 - 2a_{ij}\bar{a} + (\bar{a})^2\right]$$

$$= \sum_{i=1}^{2}\sum_{i=1}^{2}\{(a_{i1})^2 + (a_{i2})^2 - 2(a_{i1} + a_{i2})\bar{a}\}$$

$$+ \sum_{i=1}^{2}\sum_{j=1}^{2}(\bar{a})^2$$

$= (a_{11})^2 + (a_{12})^2 + (a_{21})^2 + (a_{22})^2 - 2(a_{11} + a_{12} + a_{21} + a_{22})\,\bar{a} + \sum_{i=1}^{2}\sum_{j=1}^{2}(\bar{a})^2$

$$= \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} - 2\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right)\frac{1}{2\sqrt{2}}$$

$$- \frac{1}{(2\sqrt{2})^2}\sum_{i=1}^{2}(1+1)$$

$$= 2 - \frac{2}{\sqrt{2}}\frac{1}{\sqrt{2}} - \frac{2}{2^2 \cdot 2}(1+1)$$

$= 2 - 1 - \frac{1}{2} = \frac{3}{2} = \frac{n+1}{n}$

⇒ option (A) is correct.

3. Explanation:

$$P = \frac{\underline{x}\underline{x}^{\top}}{\underline{x}^{\top}\underline{x}}$$

$$P = \begin{pmatrix} a & b \\ c & d \end{pmatrix} x = \begin{pmatrix} a \\ c \end{pmatrix}$$

$$xx^{\top} = \begin{pmatrix} a \\ c \end{pmatrix}(a \;\; c) = \begin{pmatrix} a^2 & ac \\ ac & c^2 \end{pmatrix}$$

$$\underline{x}^{\top} \cdot \underline{x} = (a \quad c)\begin{pmatrix} a \\ c \end{pmatrix} = (a^2 + c^2)$$

$$P = \frac{1}{a^2 + c^2}\begin{pmatrix} a^2 & ac \\ ac & a^2 \end{pmatrix}$$

$$P = \begin{pmatrix} a & c \\ c & c^2/a \end{pmatrix}$$

$$P = \begin{pmatrix} a & c \\ c & 1-a \end{pmatrix}$$

$$PP^{\top} = \begin{pmatrix} a & c \\ c & 1-a \end{pmatrix}\begin{pmatrix} a & c \\ c & 1-a \end{pmatrix}$$

$$\Rightarrow P^2 = \begin{pmatrix} a^2 + c^2 & c \\ c & c^2 + (1-a)^2 \end{pmatrix}$$

$$P^2 = \begin{pmatrix} a & c \\ c & 1-a \end{pmatrix} = P \neq I$$

⇒ P is idempotent, symmetric

& we take $a = 1\; c = 0$ rank of P one

⇒ P is not orthogonal.

⇒ ∴ option (B) is correct.

4. Explanation:

$$P = [P_{ij}]_{50\times 50}$$

$$P_{ij} = \min(i,j)\; i,j = 1,2,\ldots 50$$

We Take

$$p = \begin{pmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$$

$$R_2 \to R_2 - R_1$$

$$\sim \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$$\rho(p) = 2$$

$$P = \begin{pmatrix} P_{11} & P_{12} & P_{13} \\ P_{21} & P_{22} & P_{23} \\ P_{31} & P_{32} & P_{33} \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{pmatrix}$$

$$R_3 \to R_3 - R_2$$

$$R_2 \to R_2 - R_1 \qquad \sim \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\rho(p) = 3$$

$$[P]_{50\times 50} \to \rho(p) = 50$$

⇒ Option (D) is correct.

5. Explanation:

The matrix

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$$

∴ Matrix M has eigenvalue 1,4,6

∴ Which is not Nilpotent, it has linearly independents eigenvectors.

∴ M is not an elementary Matrix.

∴ M can be written as product of elementary matrices.

⇒ Option (B) is correct.

6. Explanation:

Given

$$M = \begin{pmatrix} 2 & 3 \\ 1 & -1 \end{pmatrix}$$

∴ characteristic equation is

$$\lambda^2 - (\text{trace } M) + \det(M) = 0$$
$$\text{tr}(M) = 1 \quad \det(M) = -5$$
$\Rightarrow \lambda^2 - \lambda - 5 = 0$
$\Rightarrow M^2 - M - 5I = 0$
$\Rightarrow M(M^2 - M - 5I) = 0$
i.e. $M^3 - M^2 - 5M = 0$
$\Rightarrow$ option (B) is correct.

7. Explanation: We know that the determinant of the Matrix of order $n \times n$ is equal to the product of it's n eigenvalues. The determinant is zero, Hence the product of all the eigenvalues of A will be zero, that is possible only when one of the Eigenvalues is zero.
$\Rightarrow$ Zero is eigenvalue of A.
$\Rightarrow$ option (c) is correct.

8. Explanation:
$$M = \begin{pmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{pmatrix}$$
$$|M| = \begin{vmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{vmatrix}$$
$= x(x^2 - 1) - 1(x - 1) + 1(1 - x)$
$= (x - 1)(x^2 + x - 2)$
$= (x - 1)^2(x + 2)$
For singular Matrix determinant is zero
$\Rightarrow (x - 1)^2(x + 2) = 0$
$\Rightarrow x = 1$ or $x = -2$
Therefore, there are two distinct values of x for which the given Matrix is singular.
$\Rightarrow$ Option (B) is correct.

9. Explanation:
$P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ -1 & -2 & 1 & 1 \\ 1 & 2 & -3 & -7 \\ 1 & 2 & -2 & -4 \end{bmatrix}$ Then $\rho(P) = ?$
By R.C form,
$R_2 \to R_1 + R_2$, $R_3 = R_3 - R_1$ and $R_4 = R_4 - R_1$
$P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -3 & -9 \\ 0 & 0 & -2 & -6 \end{bmatrix}$
Now $R_3 = -\frac{1}{3}R_3$ and $R_4 = -\frac{1}{2}R_4$
$P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & -1 & 3 \\ 0 & 0 & 1 & 3 \end{bmatrix}$
Now, $R_3 = R_3 - R_2$ and $R_4 = R_4 - R_2$
$P = \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
$\therefore \rho(p) = 2$
$\therefore$ Option C is correct.

10. Explanation:
Given: $p = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$ and $p^{-1} = p$
$$p^{-1} = p$$
Multiply both side by p
$$Pp^{-1} = P \cdot P$$
$$I = P^2$$
$$P^2 - I = 0.$$
$$\lambda^2 - I = 0$$
$$\lambda^2 - 0 \cdot \lambda - 1 = 0 \quad \ldots\ldots(i)$$
Now $p = \begin{bmatrix} \alpha & \beta \\ \gamma & 0 \end{bmatrix}$
$$\lambda^2 - \text{tr}(p) \cdot \lambda + \det(p) = 0$$
$$\lambda^2 - \alpha \cdot \lambda + (0 - \beta\gamma) = 0$$
$\Rightarrow \lambda^2 - \alpha \cdot \lambda - \beta\gamma = 0 \quad \ldots\ldots(ii)$
From equation (i) and (ii)
$$\alpha = 0 \text{ and } \beta\gamma = 1$$
$\therefore$ Option A is correct.

11. Explanation:
$$P\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2/5 \\ 0 \\ -2/5 \end{bmatrix}$$
$$P\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \frac{2}{5}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$
$$PV = \frac{2}{5}V$$
Multiplying both side by p
$$p^2v = \frac{2}{5}pv$$
$$p^2v = \tfrac{2}{5}\left(\tfrac{2}{5}\right)v$$
$\because pv = \frac{2}{5}v$
$$p^2\text{v} = \left(\frac{2}{5}\right)^2 v$$
$$p^3v = \left(\frac{2}{5}\right)^3 v$$
Choose first option.
(a) $(7p^2 - 2p)v = 7p^2v - 2pv$
$= 7\left(\frac{2}{5}\right)^2 v - 2\left(\frac{2}{5}\right)v$
$= \frac{2}{5}\left(7 \cdot \frac{2}{5} - 2\right)v$

$$= \frac{2}{5}\cdot\frac{2}{5}\cdot\frac{2}{5}\cdot 5v$$
$$= 5\cdot\left(\frac{2}{5}\right)^3 v$$
$$= 5p^3v$$
$$5p^3v = (7p^2-2p)v$$
$$\Rightarrow 5p^3 = 7p^2-2p$$
$$\Rightarrow p^3 = \frac{1}{5}(7p^2-2p)$$

12. Explanation: The trace of M is $\frac{1}{4}+\frac{2}{5}=\frac{13}{20}$ and the determinant of M is $\frac{1}{4}\times\frac{2}{5}-\frac{3}{4}\times\frac{3}{5}=-\frac{7}{20}$. Then, the

$$\lambda^2 - \text{trace}\,(M) + \det(M) = 0$$
$$\Rightarrow \lambda^2 - \frac{13}{20}\lambda - \frac{7}{20} = 0$$
$$\Rightarrow 20\lambda^2 - 13\lambda - 7 = 0$$

By the Cayley - Hamilton theorem, we have

$$20M^2 - 13M - 7I = 0$$

∴ Option (B) is the correct.

13. Explanation:

Let $x=\begin{pmatrix}1\\0\\0\end{pmatrix}$, $\gamma=\begin{pmatrix}0\\1\\0\end{pmatrix}$ and $z=\begin{pmatrix}0\\0\\1\end{pmatrix}$

$$x^\top = (1,0,0), Y^\top = (0,1,0)\; z^\top = (0,0,T).$$
$$\therefore Q = xx^\top + YY^\top$$
$$= \begin{pmatrix}1\\0\\0\end{pmatrix}(1,0,0) + \begin{pmatrix}0\\1\\0\end{pmatrix}(0,1,0)$$
$$\therefore Q = \begin{bmatrix}1&0&0\\0&0&0\\0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0\\0&1&0\\0&0&0\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$
$$\therefore\; Q = \begin{bmatrix}1&0&0\\0&1&0\\0&0&0\end{bmatrix}$$

∴ Option A, B, D are discarded,

∴ Option C is correct.

14. Explanation:

$$P = \begin{bmatrix}1&0&1\\0&2&1\\2&0&-1\end{bmatrix}$$
$$5p^{-1} = aI_3 + bp - p^2$$
$$\Rightarrow 6 = apI_3 + bp^2 - p^3 \text{ (multiply by } p\text{)}$$
$$\Rightarrow -p^3 + bp^2 + ap - 6I_3 = 0$$

Now, $P = \begin{bmatrix}1&0&1\\0&2&1\\2&0&-1\end{bmatrix}_{3\times3}$

For 3×3 matrix

$$C.I = \lambda^3 - (\text{tr}\,(A))\lambda^2 + [A_{11}+A_{22}+A_{33}]\lambda - |A|) = 0$$
$$\Rightarrow -\lambda^3 + [tr(A)]\lambda^2 + (-1)(A_{11}+A_{22}+A_{33})\lambda + |A| = 0$$
$$\Rightarrow \text{tr}\,(A) = b$$

now,

trace $= (1+2-1) = 2 = b$ and $a = 3$

So, option A is correct.

15. Explanation:

$if\ A\ is\ a\ skew-symmetric\ matrix.\ then$
$(I+A)\ is\ non-singulas$
$\&\ (I-A)(I+A)^{-1}\ is\ orthogonal.$
$Proof \Longrightarrow$
$$B = (I-A)(I+A)^{-1}$$
$$B^\top = [(I-A](I+A)^{-1}]^\top$$
$$= ((I+A)^{-1})^\top (I-A)^\top$$
$$= ((I+A)^1)^{-1}(I-A)^\top$$
$$= ((I+A)')^{-1}(I-A^\top).$$
$$= (I-A)^{-1}(I+A), \quad [\because A^\top = -A]$$
$\therefore (I-M)(I+M)^{-1}\ is\ always\ orthogonal.$
$\Rightarrow\ option\ (C)\ is\ correct$

16. Explanation:

$$A = \begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}$$
$$\Rightarrow A^2 = A\cdot A = \begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}\begin{pmatrix}0&1&0\\1&0&0\\0&1&1\end{pmatrix}$$
$$A^2 = \begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}$$
$$A^4 = \begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}\begin{pmatrix}1&0&0\\0&1&0\\1&1&1\end{pmatrix}$$
$$A^4 = \begin{pmatrix}1&0&0\\0&1&0\\2&2&1\end{pmatrix}$$

Continuing in this way, we get

$$A^{100} = \begin{pmatrix}1&0&0\\0&1&0\\50&50&1\end{pmatrix}$$
$$= 1+0+0+1+0+0+50+50+1$$
$$A^{100} = 103$$

∴ The sum of all the elements of A^{100} is 103

Hence, option B is correct.

17. Explanation:

$M = M_1M_2$ Where M_1 & M_2 are two 3×3 distinct of Matrix.

Result: Matrix multiplication is Linear combination of first matrix with scalar as the column of sect Matrix. or Linear combination of second Matrix with scalar as the rows of First Matrix.

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \\ \downarrow & \downarrow & \downarrow \\ v_1 & v_2 & v_3 \end{bmatrix} \quad \begin{bmatrix} i & k & l \\ m & n & 0 \\ p & q & r \end{bmatrix} \begin{matrix} \rightarrow & x_1 \\ \rightarrow & x_2 \\ \rightarrow & x_3 \end{matrix}$$

$$[v_1 \quad v_2 \quad v_3] \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = [v_1x_1 + v_2x_2 + v_3x_3]$$

∴ The rows of M are Linear combinations of rows of M_2.

∴ The coloums of M are Linear combination of coloumns of M_1

⇒ $both\ Statement\ are\ True$

⇒ $option\ (c)\ is\ correct.$

18. Explanation:

$$A = \begin{pmatrix} a & 0 \\ c & d \end{pmatrix} \quad ad = 1 \qquad c \neq 0$$

$$\text{adj } (A) = \begin{pmatrix} d & 0 \\ -c & a \end{pmatrix}$$

$$A^{-1} = \frac{1}{|A|}(\text{adj } A).$$

$$A^{-1} = \frac{1}{1}(\text{adj } A).$$

$$\Rightarrow \text{adj } A = A^{-1} \Rightarrow A = (\text{adj } A)^{-1}$$

$$\Rightarrow A^{-1} + (\text{adj } A)^{-1} = \begin{bmatrix} d & 0 \\ -c & a \end{bmatrix} + \begin{bmatrix} a & 0 \\ c & d \end{bmatrix}$$

$$\begin{bmatrix} \alpha & \beta \\ \gamma & \delta \end{bmatrix} = \begin{bmatrix} a+d & 0 \\ 0 & a+d \end{bmatrix}$$

$$(\alpha, \beta, \gamma, \delta) = (a+d, 0, 0, a+d)$$

⇒ option (A) is correct.

19. Explanation: (C)

(A) If $\text{rank}(A^5) = \text{rank}(A^6)$,

Then A^5 and A^6 span the same column space.

So $A^7 = A \cdot A^6$ won't reduce rank further.

Hence:

$\text{rank}(A^6) = \text{rank}(A^7)$ True

(B) If $\text{rank}(A) = n$, then A is invertible.

Changing even one entry (like a diagonal 1 in identity matrix to 0) can make the determinant 0.

→ Matrix becomes singular. True

(C) Given: $\text{rank}(A) = n$

Claim: $\text{rank}(A + A^T) \geq \frac{n}{2}$

But no general result guarantees this.

Counterexamples exist where $A + A^T$ is rank deficient.

Not necessarily true

(D) If $\text{rank}(A) = r < n$,

Then we can possibly increase rank to n by modifying at most $n - r$ entries.

(Though it depends on the structure, it's possible.) True

Final Answer: (C) is NOT necessarily true.

Multiple Selective Questions (MSQ)

[2019: 2 Marks]

1. Let P be an n x n non-null real skew-symmetric matrix, where n is even. Which of the following statements is (are) always TRUE?

(A) Px = 0 has infinitely many solutions, where $0 \in \mathbb{R}^n$

(B) Px = λx has a unique solution for every non-zero $\lambda \in \mathbb{R}$

(C) If $Q = (I_n + P)(I_n - P)^{-1}$, then $Q^TQ = I_n$

(D) The sum of all the eigenvalues of p is zero

[2020: 2 Marks]

2. For real constants a and b, let

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix}$$

be an orthogonal matrix. Then which of the following statements is/are always TRUE?

(A) $a + b = 0$ (B) $b = \sqrt{1 - a^2}$

(C) $ab = -\frac{1}{2}$ (D) $M^2 = I_2$

[2021: 2 Marks]

3. Let A be a 3 × 3 real matrix such that $A \neq I_3$ and the sum of the entries in each row of A is 1. Then, which of the following statements is/are necessarily TRUE?

(A) A-I_3 is an invertible matrix

(B) The set $\{x \in \mathbb{R}^3: (A-I_3)x = 0\}$ has at least two elements (x is a column vector)

(C) The characteristic polynomial, p(λ), of A+ 2 A^2 + A3 has (λ-4) as a factor

(D) A cannot be an orthogonal matrix

[2022: 2 Marks]

4. If n ≥ 2, then which of the following statements is/are true?

(A) If A and B are nxn real orthogonal matrices such that det(A) + det(B) = 0, then A + B is a singular matrix

(B) If A is an nxn real matrix such that In+A is non-singular, then $I_n + (I_n + A)^{-1} (I_n - A)$ is a singular matrix

(C) If A is an n x n real skew-symmetric matrix, then $I_n - A^2$ is a non-singular matrix

(D) If A n x n real orthogonal matrix, then $\det(A - \lambda I_n) \neq 0$ for all $\lambda \in \{x \in \mathbb{R}: x \neq \pm 1\}$

[2023: 2 Marks]

5. Let M be a 3 × 3 real matrix. If $P = M + M^T$ and $Q = M - M^T$, then which of the following statements is/are always TRUE?

(A) $\det(P^2 Q^3) = 0$

(B) $\text{trace}(Q+Q^2) = 0$

(C) $X^T Q^2 X = 0$, for all $X \in \mathbb{R}^3$

(D) $X^T PX = 2X^T MX$, For all $X \in \mathbb{R}^3$

Answer Key				
1	2	3	4	5
B,C,D	A,C	B,C	A,C,D	A,B,D

EXPLANATIONS

Multiple Selective Questions (MSQ)

1. Explanation:

For option [A]

$px = 0 \rightarrow$ infinite solution

$\Rightarrow |p| = 0$ [always]

e.g. $p = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$

$|P| = \begin{vmatrix} 0 & -1 \\ 1 & 0 \end{vmatrix} = 1$

now

$p[0] = 0 \Rightarrow$ unique solution.

∴ option A is incorrect.

B) $px = \lambda x'$ has unique solution ∀ non zero $\lambda \in \mathbb{R}$

$$px - \lambda x = 0 \Rightarrow (p - \lambda I)x = 0$$

$\Rightarrow |p - \lambda I| \rightarrow$ Cayley Hamilton theorem.

λ is an eigen value.

now λ can be zero or imaginary

so; only one possible solution [trivial]

i.e. $x = 0$

Hence option B is correct.

for option [c]

$$Q = (I + p)(I - p)^{-1}$$

$$Q^T Q = I$$

$\Rightarrow Q^T Q = ((I + p)(I - p)^{-1})'(I + p)(I - p)^{-1}$

$= [(I - p)^{-1})'(I + p)'(I + p)(I - p)^{-1}$

$\Rightarrow [(I - P)']^{-1}(I + p')(I + p)(I - P)^{-1}$

$\Rightarrow (I + p)^{-1}(I + p)(I + p)(I - p)^{-1}$

$= (I + p)^{-1}(I - p)(I + p)(I - p)^{-1}$

$= (I + \rho)^{-1}(I + p)(I - P)(I - p)^{-1}$

$= I_n$

Hence option [c] is correct.

for (D)

Sum of λ_j is $= 0$ (trace)

and $a_{ii} = 0 \quad for\ all\ i$

Hence option D is correct.

∴ option B, C and D are correct.

2. Explanation

$$M = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix}$$

$MM^T = I$ For orthogonal.

$$MM^T = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{2}a \\ \frac{1}{\sqrt{2}} & b \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ a & b \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \frac{1}{2} + a^2 = 1 \quad \Rightarrow a^2 = 1/2$$

$$\frac{1}{2} + ab = 0 \Rightarrow \qquad ab = -1/2 \Rightarrow b = -\frac{1}{2a}$$

Here

$$a + b = a + \frac{-1}{2a} = \frac{2a^2 - 1}{2a} = 0$$

$\Rightarrow a + b = 0$ & $ab = -1/2$

$\Rightarrow$ option (A) & (C) are correct.

3. Explanation:

Sum of each row is 1

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$A - I \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = 0$$

$= 0$

nullity $(A - I) \geqslant 1$

$A - I$ is not full rank, as $\eta(A - I) \neq 0$.

so, $A - I$ is not invertible.

For option B :

$$A \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

Also,

$$A \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \end{pmatrix}$$

and so on

So, $\{x, (A - I_3) \cdot x = 0\}$ has atleast two elements.

For option (c): 1 is an eigen value of A; as

$$A\begin{pmatrix}1\\1\\1\end{pmatrix} = 1\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$\therefore$ eigen values of $2A^2 = 2 \times 1 = 2$

eigen values of $A^3 = 1$

So, one eigen value of $A + 2A^2 + A^3 = 1 + 2 + 1 = 4$ so $\lambda = 4$ is an eigen value of $A + 2A^2 + A^3$.

$\therefore$ The Charactestic polynomial will have $(\lambda - 4)$ as a factor.

For option D :

As $\eta(A - I)$ has atleast two elements

$A - I$ is not invertible $\Rightarrow A$ is not invertible

$\Rightarrow A^{-1}$ cannot exist

$\Rightarrow$ A cannot be orthogonal

$\therefore$ option [B and [C] are correct.

4. Explanation:

If $n \geq 1$

(A) A and B are $n \times n$ orthogonal matrices such that $\det(A) + \det(B) = 0$ then $A + B$ is singular matrices.

$\Rightarrow A^\top A = AA^\top = I$

$$B^\top B = BB^\top = I$$

if $A + B$ is singular $\exists\ v \in \mathbb{R}^n$ such that;

$$V: (A + B)v = 0$$

$\Leftrightarrow V: A^\top(A + B)v = 0 \Leftrightarrow A^\top BV = -v$

we have to check if $A^\top B$ has eigen value -1 .

$$(A^\top B)^\top A^\top B = B^\top \mathrm{A} A^\top B = B'B = I$$

$A^\top B$ is orthogonal with $\det(A^\top B) = \det(A) \cdot \det(B)$

$= -\{\det(A)\}^2$

$= -1$ $\qquad$ $\{\text{as } \det(A) + \det(B) = 0\}$

Now, $A^\top B$ is orthogonal, and determinant of $A^\top B = -1$

Let $M = A^\top B$, can M, an orthogonal matrix with $\det(M) = -1$ have an eigen value -1?

$$Mv = \lambda v$$

$$|Mv|^2 = (Mv)^\top(Mv) = \frac{v^\top M^\top Mv}{I} = v^\top v$$

$$|Mv|^2 = |\lambda v|^2$$

$$v^\top v = \lambda^2 v^\top v$$

$\Rightarrow |v|^2(|\lambda|^2 - 1) = 0 \Rightarrow |\lambda|^2 = 1 \Rightarrow |\lambda| = 1$

$\Rightarrow \lambda = \pm 1$

[Also, λ can be complex and occur in pairs $(\lambda_1\lambda_1^*)$, (λ_2, λ_2^*) ... Their product : $|\lambda_1||\lambda_2| \ldots |\lambda_n| = 1$]

But the product is -1, so f at least one $\lambda = -1$.

$\Rightarrow \lambda = -1$ is an eigenvalue of $A^\top B$.

$A^\top B$ has -1 as eigenvalue.

$A^\top BV = -v$

$(I + A^\top B)v = 0 \Rightarrow (A^\top A + A^\top B)v = 0$

$\Rightarrow A^\top(A + B)v = 0$

$\Rightarrow A \cdot A^\top(A + B)v = 0 \Rightarrow (A + B)\vec{v} = 0.$

as $\vec{v} \neq 0, |A + B| = 0$

For option B

: A is an $n \times n$ real matrix such that $In + A'$ is non singular, then $I_n + (I_n + A)^{-1}(I_n - A)$ is a singular matrix.

$I_n + A$ is non singular

$$Av = \lambda v$$

$$(I + A)v = (1 + \lambda)v$$

$$(I + A)^{-1}v = (1 + \lambda)^{-1}v$$

$$(I - A)v = (1 - \lambda)v$$

$$\therefore (I + (I + A)^{-1}(I - A))v = \left(1 + \frac{1 - \lambda}{1 + \lambda}\right)v$$

$$= \left(\frac{2}{1 + \lambda}\right)v$$

$\Rightarrow \dfrac{2}{1 + \lambda} \neq 0$

So, $I_{n+}\ (I_n + A)^{-1}\ (I_n - A)$ has non-zero Eigen valued

$\Rightarrow$ It is not a singular matrix.

for option (c):

A is nan real skew symmetric,

$I_n - A^2$ is a nonsingular matrix.

$$Av = \lambda v$$

$$(I - A^2)v = (1 - \lambda^2)v$$

$I - A^2$ is singular if $1 - \lambda^2 = 0$

$\Rightarrow \lambda^2 = 1$

$\Rightarrow \lambda = \pm 1$

But A is skew symmetric

$\Rightarrow$ It has purely imaginary or 0 as its eigen values.

So, λ cannot be ± 1.

$\Rightarrow I - A^2$ is non singular.

For option D:

A is $n \times n$ real orthogonal matrix. then $|A - \lambda I_n| \neq 0\ \forall\ \lambda \in \{x \in \mathbb{R}\,; x \neq \pm 1\}$

A is real orthogonal

$\Rightarrow |A| = \pm 1$

$|A - \lambda I| = |A^\top - \lambda I|$

if $\lambda = 1$

$(A - I)v = (\lambda' - 1)$v where λ' is an eigen value of A

$$A^\top(A - I)v = (I - A^\top)v = \left(1 - \frac{1}{\lambda'}\right)v$$

if $\lambda' = 1, (A - I)$ and $(I - A^\top)$ are singular.

(as they have 0 as Eigen value)

similarly, if $\lambda = -1$.

$(A + I)$ and $(A^\top + I)$ are singular

$\Rightarrow |A - \lambda I_n| = 0$ if $= \lambda \pm 1$

for any other values of $\lambda, |A - \lambda I_n| \neq 0$

Hence option A, C and D are correct.

5. Explanation

$$M \text{ be a } 3 \times 3 \text{ real matrix } 3 \times 3$$

$$P = M + M^\top \quad Q = M - M^\top$$

$$M = \frac{M + M^\top}{2} + \frac{M - M^\top}{2} = \frac{P + Q}{2}$$

$$P \rightarrow \text{ symmetric } (p^\top = p).$$

$$Q \rightarrow \text{ Skew symmetric } (Q^\top = -Q)$$

(A)

$$\det (P^2 Q^3) = 0$$

$$Q \rightarrow \text{ Skew symmetric}$$

$$|Q| = 0 \text{ by the determinant of skew} - \text{ symmetric}$$

$$\text{Matrix of odd order always zero.}$$

$\therefore$ option (A) is correct

(B)

$$\text{tr } (Q + Q^2) = 0$$

$$\text{Every diagonal element of skew} - \text{symmetric Matrix is Zero.}$$

$\therefore$ option (B) is correct.

(C)

$$X^\top Q^2 x = for\ all\ x \in \mathbb{R}^3$$

$$\text{Eigen value of Skew} - \text{symmetric Matrix} \rightarrow 0, \pm ia$$

$$Q \rightarrow 0, a^2, -a^2$$

$X^\top Q^2 X \neq 0$ for all $x \in \mathbb{R}^3$

$\therefore$ option (c) is wrong.

(D)

$$X^\top PX = 2x^T MX. \quad for\ all\ x \in R^3.$$

$$M = I \quad P = M + M^\top \quad Q = M - M^\top$$

$$M^\top = I \quad P = 2I \quad Q = 0$$

$$X^\top PX = X^\top (2I) X = 2X(I)X$$

$$= 2x^\top X \qquad \Rightarrow X^\top PX = 2X^\top MX.$$

$$2X^\top MX = 2X^\top (I) X = 2X^\top X$$

$\Longrightarrow$ option (D) is correct.

$\Rightarrow$ option $(A), (B), \& (D)$ correct.

Numerical Answer Type (NAT)

[2017: 1 Mark]

1. For j = 1,2,..., 5, let P_j be the matrix of order 5 x 5 obtained by replacing the j[th] column of the identity matrix of order 5×5 with the column vector v = $(5\ 4\ 3\ 2\ 1)^t$. Then the determinant of the matrix product $P_1\ P_2\ P_3\ P_4\ P_5$ is _____ .

[2017: 2 Marks]

2. Let a unit vector v = ($V_1\ V_2\ V_3$) be such that Av=0 where

$$A = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & \frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

Then the value of √6 ($|v_1| + |v_2| + |V_3|$) equals _____ .

[2018: 2 Marks]

3. Let M = $\sum_{i=1}^{4} X_i X_i^T$ where X_1^T = [1 -1 1 0], X_2^T = [1 1 0 1], X_3^T = [1 3 1 0] and X_4^T = [1 1 1 0]. Then the rank of M equals _____

[2019: 1 Mark]

4. If $\begin{bmatrix} \frac{\sqrt{5}}{3} & \frac{-2}{3} & c \\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d \\ a & b & 1 \end{bmatrix}$ is a real orthogonal matrix, then $a^2 + b^2 + c^2 + d^2$ equals _____

[2019: 2 Marks]

5. If $ad - bc = 2$ and $ps - qr = 1$, then the determinant of $\begin{bmatrix} a & b & 0 & 0 \\ 3 & 10 & 2p & q \\ c & d & 0 & 0 \\ 2 & 7 & 2r & s \end{bmatrix}$ equals _____

[2020: 1 Mark]

6. The rank of the matrix

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 2 \\ 2 & 5 & 6 & 4 \\ 2 & 6 & 8 & 5 \end{bmatrix}$$

is _____

[2020: 2 Marks]

7. Consider the matrix

$$M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 1 & 4 \end{bmatrix}$$

Let P be a non-singular matrix such that $P^{-1}MP$ is a diagonal matrix. Then the trace of the matrix $P^{-1}M^3P$ equals _____ .

[2021: 1 Mark]

8. Let $M = \begin{pmatrix} 5 & -6 \\ 3 & -4 \end{pmatrix}$ be a 2 × 2 matrix. If $\alpha = \det(M^4 - 6I_2)$, then the value of α^2 is _____.

[2022: 1 Mark]

9. Let A be a 3× 3 real matrix such that det(A) = 6 and adj A = $\begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix}$ where adj A denotes the adjoint of A. Then the trace of A equals _____ (round off to 2 decimal places)

Answer Key								
1	2	3	4	5	6	7	8	9
120	4	3	0	-4	3	134	2500	3.5

EXPLANATIONS

Numerical Answer Type (NAT)

1. Explanation: Since the determinant of the product of the matrix is equal to the product of their determinants, we have

$$|P_1| = 5\cdot\begin{vmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{vmatrix} + 4\begin{vmatrix} 0&0&0&0&0\\0&1&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{vmatrix} +$$

$$3\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&0&0\end{vmatrix} + 2\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{vmatrix} +$$

$$1\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{vmatrix}$$

$$= 5 \times 1 = 5$$

$$|P_2| = 4\cdot\begin{vmatrix} 1&0&0&0&0\\0&1&0&0&0\\0&0&1&0&0\\0&0&0&1&0\\0&0&0&0&1\end{vmatrix} + 5\begin{vmatrix} 1&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\end{vmatrix}$$

$$+ 3\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&1&0&0\\0&0&0&0&0\\0&0&0&&0\end{vmatrix}$$

$$+2\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&1&0\\0&0&0&0&0\end{vmatrix} + 1\begin{vmatrix} 0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&1\end{vmatrix}$$

$$= 4$$

Similarly, $|P_3| = 3$, $|P_2| = 2$, $|P_1| = 1$

So $|P_1P_2P_3P_4P_5| = |P_1||P_2||P_3||P_4||P_5|$

$$= 120$$

2. Explanation:

$$A = \begin{pmatrix} \frac{5}{6} & -\frac{1}{3} & -\frac{1}{6} \\ -\frac{1}{3} & \frac{1}{3} & \frac{-1}{3} \\ -\frac{1}{6} & -\frac{1}{3} & \frac{5}{6} \end{pmatrix}$$

we have given that $AV = 0$.

$$\Rightarrow \frac{1}{6}\begin{pmatrix} 5 & -2 & -1 \\ -2 & 2 & -2 \\ -1 & -2 & 5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Hence, the system of linear equation is given by

$$5v_1 - 2v_2 - v_3 = 0 \ldots..(1)$$

$$-2v_1 + 2v_2 - 2v_3 = 0 \ldots\ldots(2)$$

$$-v_1 - 2v_2 + 5v_3 = 0 \ldots..(3)$$

solving equation (1), (2) and (3) then we get,

$$v_1 = v_3 \text{ and } v_2 = 2v_3$$

since $v = (v_1, v_2, v_3)^T$ is a unit vector.

i.e., $v_1^2 + v_2^2 + v_3^2 = 1$.

$\Rightarrow |v_1| = \frac{1}{\sqrt{6}}$, $|v_2| = \frac{2}{\sqrt{6}}$ and $|v_3| = \frac{1}{\sqrt{6}}$

Hence the value of

$$\sqrt{6}(|v_1| + |v_2| + |v_3|) = \sqrt{6}\left(\frac{1}{\sqrt{6}} + \frac{2}{\sqrt{6}} + \frac{1}{\sqrt{6}}\right) = 4$$

3. Explanation:

$$M = \sum_{i=1}^{4} X_i X_i^T = [X_1\ X_2\ X_3\ X_4]\begin{bmatrix} X_1^T \\ X_2^T \\ X_3^T \\ X_4^T \end{bmatrix} = AA^T,$$

$$A = [X_1\ X_2\ X_3\ X_4] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 4 & 2 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (R_2 \to R_2 + R_1,\ R_3 \to R_3 - R_1)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad (R_2 \to \tfrac{1}{2}R_2)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & -2 & -1 \end{bmatrix} \quad (R_3 \to R_3 + R_2, R_4 \to R_4 - R_2)$$

$$= \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (R_4 \to R_4 + R_3)$$

$\therefore$ rank(M)=rank(AA^T)=rank(A)=3

4. Explanation:

$$A=\begin{bmatrix}\frac{\sqrt{5}}{3} & -\frac{2}{3} & c\\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d\\ a & b & 1\end{bmatrix}$$

we know that orthogonal property:

$$AA^{\mathrm{t}}=I=A^{t}A$$

$$AA^{\mathrm{t}}=$$

$$\begin{bmatrix}\frac{\sqrt{5}}{3} & -\frac{2}{3} & c\\ \frac{2}{3} & \frac{\sqrt{5}}{3} & d\\ a & b & 1\end{bmatrix}\begin{bmatrix}\frac{\sqrt{5}}{3} & \frac{2}{3} & a\\ -\frac{2}{3} & \frac{\sqrt{5}}{3} & b\\ c & d & 1\end{bmatrix}=\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{bmatrix}\frac{5}{9}+\frac{2^2}{3^2}+c^2 & \frac{2\sqrt{5}}{9}-\frac{2\sqrt{5}}{9}+d^2 & \frac{a\sqrt{5}}{3}-\frac{2b}{3}+c\\ \frac{2\sqrt{5}}{9}-\frac{2\sqrt{5}}{9}+cd & \frac{4}{9}+\frac{5}{9}+d^2 & \frac{2}{3}a+\frac{b\sqrt{5}}{3}+d\\ \frac{a\sqrt{5}}{3}-\frac{b2}{3}+c & \frac{2}{3}a+\frac{\sqrt{5}}{3}b+d & a^2+b^2+1\end{bmatrix}=$$

$$\begin{bmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{bmatrix}$$

compare this matrix by R. H. S

$\therefore \frac{5}{9}+\frac{4}{9}+c^2=1$

$\Rightarrow c^2=0$

$$d^2+\frac{9}{9}=1$$

$$d^2=0$$

$$a^2+b^2+1=1$$

$$a^2+b^2=0$$

Hence $a^2+b^2+c^2+d^2=0$

5. Explanation:

Let $A=\begin{bmatrix}a & b & 0 & 0\\ 3 & 10 & 2p & q\\ c & d & 0 & 0\\ 2 & 7 & 2r & 5\end{bmatrix}$

$$ad-bc=2,\quad ps-qr=1$$

$$|A|=a\begin{vmatrix}10 & 2p & q\\ d & 0 & 0\\ 7 & 2r & s\end{vmatrix}-b\begin{vmatrix}3 & 2p & q\\ c & 0 & 0\\ 2 & 2r & s\end{vmatrix}$$

$=a(-d(2ps-2rq))-b(-c(2ps-2\mathrm{rq}))$

$=a(-d(2))-b(-c(2)).$

$=-2ad+2bc$

$=-2(ad-bc)$

$=-2\times 2=-4$

$\therefore |A|=-4$

6. Explanation:

$$M=\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 2 & 3 & 2\\ 2 & 5 & 6 & 4\\ 2 & 6 & 8 & 5\end{bmatrix}$$

$R_2\to R_2-R_1$

$R_3\to R_3-2R_1$

$R_4\to R_4-2R_1$

$$\sim\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 1\\ 0 & 3 & 4 & 2\\ 0 & 4 & 6 & 3\end{bmatrix}\begin{matrix}R_3\to R_3-3R_2\\ R_4\to R_4-4R_2\end{matrix}$$

$$\sim\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & -2 & -1\\ 0 & 0 & -2 & -1\end{bmatrix}$$

$R_4\to R_4-R_3$

$$M=\begin{bmatrix}1 & 1 & 1 & 1\\ 0 & 1 & 2 & 1\\ 0 & 0 & -2 & -1\\ 0 & 0 & 0 & 0\end{bmatrix}$$

$\Rightarrow \mathrm{Rank}\,(M)=3$

7. Explanation:

$$M=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 2\\ 0 & 1 & 4\end{bmatrix}$$

if A & P are square matrix of same type & P is invertible then A & $P^{-1}AP$ have same eigen values.

$$M=\begin{bmatrix}1 & 0 & 0\\ 0 & 3 & 2\\ 0 & 1 & 4\end{bmatrix}$$

$$(1+\lambda)(\lambda^2-7\lambda+10)$$

$$\partial=1,2,5$$

Now $P^{-1}MP$ is diagonal matrix say D & eigen values of D are its diaganal entries

$$So,\ P^{-1}MP=\begin{bmatrix}1 & 0 & 0\\ 0 & 2 & 0\\ 0 & 0 & 5\end{bmatrix}=D.$$

Eigenvalues of $P^{-1}M^3P=$ Eigen values D^3

$\Rightarrow$ Trace $=1^9+2^3+5^3$

$=134$

8. Explanation:

$M=\begin{pmatrix}5 & -6\\ 3 & -4\end{pmatrix}$

$\alpha=det(M^4-6I_n)$

$M=\begin{pmatrix}5-\lambda & -6\\ 3 & -4-\lambda\end{pmatrix}$

$=-20-5\lambda+4\lambda+\lambda^2+18=0$

$=\lambda^2-\lambda-2=0$

$$\lambda=-1,2$$

eigen values of M are $-1,2$

$$M \to -1,2$$
$$M^4 \to 1,16$$

now for $M = -1$ then

$$M^4 - 6 = 1 - 6 = -5$$

For $M = 16$ then

$M^4 - 6I = 16 - 6 = 10$

$\therefore\ |M^4 - 6I| =$ product of eigen values.

$\Rightarrow\ |M^4 - 6I| = -5 \times 10$

$= -50.$

$\therefore\ \alpha = \det(M^4 - 6I)$

$$\alpha = -50$$

$\therefore\ \alpha^2 = (-50)^2$

$\Rightarrow\ \alpha^2 = 2500$

9. Explanation:

$$\text{adj } A = \begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$= 1(7-1) + 1(5+1) + 2(5+7)$

$= 6 + 6 + 2(12)$

$= 12 + 24 = 36$

Given: $\det(A) = 6$

we know that,

$$A^{-1} = \frac{1}{|A|} adjA = \frac{1}{6}\begin{pmatrix} 1 & -1 & 2 \\ 5 & 7 & 1 \\ -1 & 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1/6 & -1/6 & 2/6 \\ 5/6 & 7/6 & 1/6 \\ -1/6 & 1/6 & 1/6 \end{pmatrix} = B$$

$$\Rightarrow |B| = \begin{vmatrix} 1/6 & -1/6 & 1/6 \\ 5/6 & 7/6 & 1/6 \\ -1/6 & 1/6 & 1/6 \end{vmatrix}$$

$$|B| = \frac{1}{6}\left[\frac{7}{36} - \frac{1}{36}\right] - \left(\frac{-1}{6}\right)\left[\frac{5}{36} + \frac{1}{36}\right] + \frac{2}{6}\left(\frac{5}{36} + \frac{7}{36}\right)$$

$$= \frac{1}{216}[6 + 6 + 24]$$

$$= \frac{36}{216} = \frac{1}{6}$$

$$1B1 = \frac{1}{6}$$

Now

$$\text{adj } B \Rightarrow \begin{pmatrix} \frac{6}{36} & \frac{6}{36} & \frac{12}{36} \\ \frac{-3}{36} & \frac{3}{36} & 0 \\ \frac{-15}{36} & \frac{-9}{36} & \frac{12}{36} \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \frac{6}{36} & -\frac{6}{36} & \frac{12}{36} \\ \frac{3}{36} & \frac{3}{36} & 0 \\ -\frac{15}{36} & \frac{9}{36} & \frac{12}{36} \end{pmatrix}^{T} \Rightarrow \begin{pmatrix} \frac{6}{36} & \frac{3}{36} & -\frac{15}{36} \\ -\frac{6}{36} & \frac{3}{36} & \frac{9}{36} \\ \frac{12}{36} & 0 & \frac{12}{36} \end{pmatrix}$$

$$\text{Adj } B \Rightarrow \begin{pmatrix} 116 & 1112 & -5112 \\ -116 & 1112 & 114 \\ 113 & 0 & \frac{1}{3} \end{pmatrix}$$

$$\text{now } B^{-1} = 6\begin{bmatrix} 1/6 & 1/12 & -5/12 \\ -1/6 & 1/12 & 1/4 \\ 1/3 & 0 & 1/3 \end{bmatrix}$$

$$B^{-1} = \begin{bmatrix} 1 & 1/2 & -5/2 \\ -1 & 1/2 & 3/2 \\ 2 & 0 & 2 \end{bmatrix} = adjA$$

$\therefore$ trace$(\text{adj } A) = 1 + \frac{1}{2} + 2 = \frac{3}{2} + 2 = \frac{3+4}{2} = \frac{7}{2} = 3.5$

trace (adj A) $= 3.5$