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REAL ANALYSIS

Unit 1.1: Point Set Topology

[CUET PG]

[CUET PG 2022]

1. Let A and B be two non-empty subsets of R . If $\underline{A}$, A° , and A' denote closure, interior and set of limit points of A respectively, then

- (1) $A^\circ \cap \underline{B}$ is open (2) $A^\circ \cap B'$ is open
(3) $\underline{A} \cap B'$ is closed (4) $\underline{A} \cap B$ is closed

Answer Key

1

3

[CU-CET]

[CUCET 2016]

1. Consider the set $S = \left\{x \in R : \frac{2x+1}{x+2} < 1\right\}$, where R is the set of reals. Determine which one of the following statements about S is correct.

- (1) S is bounded below but not above and $\inf S = -2$
(2) S is bounded above but not below and $\sup S = 1$
(3) S is bounded both below and above with $\inf S = -2, \sup S = 1$
(4) S is neither bounded below nor above

[CUCET 2016]

2. Consider the set $S = \left\{\frac{mn}{1+m+n} : m, n \text{ are natural numbers}\right\}$. Then determine which one of the following statements is correct.

- (1) S is a bounded set
(2) S is bounded below with $\inf S = \frac{1}{3}$ but not bounded above
(3) S is bounded above with $\sup S = 1$ but not below
(4) S is neither bounded below nor above

[CUCET 2016]

3. Let $X = \{x \in R : x \neq 1, 2, \dots, 100\}$ be a subset of R . Determine which one of the following statements is true

- (1) Integers $1, 2, \dots, 100$ are the only limit points of X
(2) No limit point of X lies between 1 and 100
(3) X is a closed subset of R
(4) X is an open subset of R

[CUCET 2016]

4. Let $X = \left\{\frac{1}{n} : n \in Z, n \neq 0\right\}$ be a subset of R (Z is the set of all integers). Determine which one of the following properties of X is true.

- (1) X is a bounded set
(2) X is an open subset of R
(3) X is a closed subset of R
(4) X has no limit point in R

[CUCET 2016]

5. The set Z of all integers

- (1) is an open subset of R
(2) is a closed subset of R
(3) is a compact subset of R
(4) has infinitely many limit points in R

[CUCET 2016]

6. Determine which one of the following subsets of R is connected.

- (1) $(-\infty, 0) \cup (0, \infty)$
(2) The set Q of all rational numbers
(3) $\bigcup_{n=1}^{\infty} (-n, n)$
(4) $R \setminus Z$

[CUCET 2016]

7. Which one of the following statements is true

- (1) Every closed interval in R is homeomorphic to R
(2) Every open interval of the type (a, b) is homeomorphic to R
(3) Every interval of the type $[a, b]$ is homeomorphic to R
(4) Every interval of the type $(a, b]$ is homeomorphic to R

[CUCET 2017]

8. The set $U = \left\{x \in R : \sin \sin x = \frac{1}{2}\right\}$ is

- (1) open
(2) closed
(3) both open and closed
(4) neither open nor closed

[CUCET 2021]

9. Let Q denotes the complement of the set of rational numbers Q . The set of all boundary points of Q in the set of real numbers R is

- (1) Q^c (2) R
 (3) ϕ (4) Q

[CUCET 2021]

10. The set $U = \{x \in \mathbb{R} \mid \sin x = \frac{1}{2}\}$ is

- (1) closed
 (2) open
 (3) neither open nor closed
 (4) both open and closed

Answer Key									
1	2	3	4	5	6	7	8	9	10
3	2	4	1	2	3	2	2	2	1

[PRACTICE QUESTIONS]

- Let A be a closed subset of $\mathbb{R}$, $A \neq \phi$, $A \neq \mathbb{R}$. Then A is
 - the closure of the interior of A .
 - a countable set.
 - a compact set.
 - not open.
- Let A and B be two sets having m and n many elements respectively, where $m < n$; $m, n \in \mathbb{N}$. The number of injective functions possible from A to B is:
 - m^n
 - ${}^nP_m$
 - ${}^nC_m$
 - n^m
- The set of rational numbers has:
 - Least upper bound property
 - Greatest lower bound property
 - Archimedean property
 - None of the above
- Which of the following statement is not true?
 - $\sqrt{2}$ is an irrational number
 - $\sqrt{2}$ is not an algebraic number
 - $\sqrt{2}$ is not a transcendental number
 - $\sqrt{2}$ is not a rational number
- Which set is uncountable?
 - The set of positive primes
 - The set of integers
 - The set of rational numbers
 - The set of irrational numbers in $[0, 1]$
- An algebraic number is a root of a polynomial whose coefficients are rational. The set of algebraic number is
 - finite
 - countably infinite
 - uncountable
 - none of these
- Which one of the following does not imply $= 0$?
 - For all $\varepsilon > 0, 0 \leq \alpha < \varepsilon$
 - For all $\varepsilon > 0, -\varepsilon < \alpha < \varepsilon$
 - For all $\varepsilon > 0, \alpha < \varepsilon$
 - For all $\varepsilon > 0 - \varepsilon, < \alpha < \varepsilon$
- Let S and T be subsets of $\mathbb{R}$. Select the incorrect statement:
 - $(\text{int } S) \cap (\text{int } T) = \text{int}(S \cap T)$
 - $(\text{int } S) \cup (\text{int } T) \subset \text{int}(S \cup T)$
 - $\bar{S}$ is closed in $\mathbb{R}$
 - $\bar{T}$ is the largest closed set containing T
- Choose the correct statements
 - $\bigcup_{n=1}^{\infty} [1/n, 2] = [0, 2]$
 - $\bigcup_{n=1}^{\infty} [1/n, 2] = (0, 2]$
 - $\bigcap_{n=1}^{\infty} (1/n, 2] = (1, 2]$
 - $\bigcap_{n=1}^{\infty} \left(1 - \frac{1}{n}, 2\right] = (1, 2]$
- If $\mathbb{Q} \subset A \subset \mathbb{R}$, which of the following must be true?
 - If A is open, then $A = \mathbb{R}$
 - If A is closed, then $A = \mathbb{R}$
 - If A is uncountable, then A is closed
 - If A is countable, then A is closed
- Which one of the following set is not countable?
 - $\mathbb{N}^r$, where $r \geq 1$ and $\mathbb{N}$ is the set of natural numbers
 - $\{0, 1\}^{\mathbb{N}}$, the set of all the sequences which takes values 0 and 1.
 - $\mathbb{Z}$, set of integers
 - $\sqrt{2}\mathbb{Q}$, $\mathbb{Q}$ is set of rational numbers.
- Let X be a countably infinite subset of $\mathbb{R}$ and A be a countably infinite subset of X . Then the set $X \setminus A = \{x \in X \mid x \notin A\}$
 - is empty
 - is a finite set.
 - can be a countably infinite set.
 - can be an uncountable set

13. The subset $A = \{x \in \mathbb{Q} : x^2 < 4\}$ of $\mathbb{R}$ is
 (1) bounded above but not bounded below
 (2) bounded above and $\sup A = 2$.
 (3) bounded above but does not have a supremum.
 (4) not bounded above.
14. The subset $A = \{x \in \mathbb{Q} : -1 < x < 0\} \cup$ of $\mathbb{R}$ is
 (1) bounded, infinite set and has a limit point in $\mathbb{R}$
 (2) unbounded, infinite set and has a limit point in $\mathbb{R}$
 (3) unbounded, infinite set and does not have a limit point in $\mathbb{R}$
 (4) bounded, infinite set and does not have a limit point in $\mathbb{R}$
15. Let

$$S = \prod_{n=1}^{\infty} \left[-\frac{1}{2}, 1 + \frac{1}{n} \right]$$
 then S equals [DU-2017]
 (1) $[0,1]$ (2) $(0,1]$
 (3) $(0,1)$ (4) $[10,1)$
16. Consider the sets

$$S = \left\{ \frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime} \right\} T = \{x^2 : x \in \mathbb{R}\}$$
 Then
 (1) $\sup(S \cap T) = 1$
 (2) $\sup S = 1$ and $\inf T = 0$
 (3) $\sup S = \frac{1}{2}$ and $\inf T = 0$
 (4) $\inf(S \cup T) = \frac{1}{2}$
17. Which of the following sets is in one-to-one correspondence with $\mathbb{N}$
 (I) $\left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \dots \right\}$
 (II) $\{ \dots, 3, -3, -2, -1, 0, 1, 2, 3, \dots \}$
 (III) $\left\{ \frac{p}{q} : p, q \in \mathbb{N}, q \neq 0 \right\}$
 (IV) $\left\{ \frac{p}{q} : p, q \in \mathbb{N} \right\}$
 (1) (I) and (II) (2) (I), (II) and (II)
 (3) (I) and (IV) (4) All of above
18. Let $X = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y \in \mathbb{R} \setminus \mathbb{Q}\}$. Where $\mathbb{Q}$ is the set of rationals. Then
 (1) X is an open and dense subset of $\mathbb{R}^2$
 (2) X is an open but not dense subset of $\mathbb{R}^2$
 (3) X is not an open but a dense subset of $\mathbb{R}^2$
 (4) X is neither an open nor a dense subset of $\mathbb{R}^2$
19. Let $S = \{x \in \mathbb{R} : 3 - x^3 > 0\}$. Then
 (1) S is bounded above and 3 is the least upper bound of S .
 (2) S is bounded above and does not have a least upper bound in $\mathbb{R}$
 (3) S is bounded above and does not have a least upper bound in $\mathbb{Q}$, the set of rational members.
 (4) S is not bounded above.
20. Let P be the set of all the polynomials with rational coefficients and S be the set of all sequences of natural numbers. Then which one of the following statement is true?
 (1) S is countable but is not
 (2) Both the sets P and S are uncountable
 (3) Both the sets P and S are countable.
 (4) P is countable but S is not
21. Let $S = \bigcap_{n=1}^{\infty} \left[2 - \frac{1}{n}, 3 + \frac{1}{n} \right]$.
 Then S equals
 (1) $(2,3]$ (2) $[2,3]$
 (3) $[2,3)$ (4) $(2,3)$.
22. Let $S = \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n} \right]$ then
 (1) $\limsup a_n = +\infty, \liminf a_n = -1$
 (2) $\limsup a_n = +\infty, \liminf a_n = 0$
 (3) $\limsup a_n \neq +\infty, \liminf a_n = -\infty$
 (4) $\limsup a_n = 1, \liminf a_n = -1$
23. Let $S = \bigcap_{n=1}^{\infty} \left[0, \frac{1}{n} \right]$. Then which one of the following statement is true?
 (1) $\inf S > 0$
 (2) $\sup S = 1$ and $\inf S = 0$
 (3) $\sup > 0$
 (4) $\sup S = \inf S = 0$
24. Let $S = \{x \in \mathbb{R} : x^2 \leq 5\} \cap \mathbb{Q}$ which of the following statements is true about S ?
 (1) S is bounded above and $\sup S \in \mathbb{Q}$
 (2) S is bounded above and $\sup S \notin \mathbb{Q}$
 (3) S is a closed interval.
 (4) S is an open interval

25. The set of all real numbers x for which there is some positive real number y such that $x < y$ equal to
 (1) $\mathbb{R}$
 (2) The set of all negative real numbers
 (3) $\{0\}$
 (4) None of the above
26. Let $X \subset \mathbb{N}$ be a nonempty finite set and $\mathbb{N}$ be an infinite set. Define $A = \{x - y : x \in X \text{ and } y \in \mathbb{N}\}$. Then
 (1) $\inf(A) = -\infty$ and $\sup(A) = \infty$.
 (2) $\inf(A) = -\infty$ and $\sup(A) < \infty$.
 (3) $\inf(A) > -\infty$ and $\sup(A) = \infty$.
 (4) $\inf(A) = -\infty$ and $\sup(A) < \infty$.
27. Let X be a nonempty set and $A, B \subset X$. If $(A \cup B) \setminus (A \cap B)$ is a finite set, then,
 (1) Both A and B are finite sets.
 (2) At least one of A, B is a finite set
 (3) X is a finite set
 (4) None of the above
28. Let X be a set with $2m+1$ elements, where $m \geq 4$. The number of subsets of X having at least $m+1$ elements is
 (1) $2^{2m} + 1$ (2) $2^{2m} - 1$
 (3) 2^{2m} (4) $2^m + 1$
29. Let $A \Delta B$ denote the symmetric difference of two sets : $A \Delta B := (A \setminus B) \cup (B \setminus A)$ which of the following is true.
 (1) $A \cap (B \Delta A) = \emptyset$ always hold true
 (2) $(A \cup (A \Delta B)) \cap B \neq B$ always hold true .
 (3) $A \Delta B = (A \cup B) \cap (A^c \cap B^c)$ always holds true
 (4) None of the above is true
30. For subsets T_1 and T_2 of $\mathbb{R}$,
 we define $T_1 + T_2 := \{t_1 + t_2 | t_1 \in T_1, t_2 \in T_2\}$.
 Let
 $S_1 := (\mathbb{Q} \cap [0,1]) + \{\sqrt{2}\}$ and
 $S_2 := (\mathbb{Q} \cap [1,2]) + \{\sqrt{3}\}$. Let $S = S_1 \cup S_2$.
 Which one of the following is true?
 (1) There exists a one to one map from $[0,1]$ to S
 (2) There exists an onto map from S to $[0,1]$
 (3) $\#S_1 = \#S_2$
 (4) None of the above
31. Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be a continuous map and $S = \left\{ \pi + \frac{(-1)^n}{2n+1} | n \in \mathbb{N} \right\} \cup (0,1)$. Which one of the following is necessarily true?
 (1) $f((0,1))$ is an open set
 (2) $f(\bar{S})$ is compact
 (3) $f(S)$ is compact
 (4) $f(S)$ is closed but not compact
32. Which of the following is a compact subset of $\mathbb{R}$?
 (1) $\left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup \{0\} \cup (0, -]$
 (2) $\left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup (0, -]$.
 (3) $\{1,2,3\} \cup \left\{ 6 + \frac{1}{n} : n \in \mathbb{N} \right\}$.
 (4) $\left\{ -\frac{1}{n} : n \in \mathbb{N} \right\} \cup (0,2]$
33. Which of the following sets has/have the same cardinality as $\mathbb{R}$?
 A. The set of all real sequences
 B. The set of all continuous functions from $\mathbb{R}$ to $\mathbb{R}$
 C. The set of all functions from $\mathbb{R}$ to the two-element set $\{0,1\}$
 (1) A only (2) B only
 (3) A and B only (4) C only
34. Let $f: \mathbb{R}^3 \rightarrow \mathbb{R}$ be a continuous function such that $f(\alpha v) = f(v)$ for all $\alpha \in \mathbb{R}, \alpha > 0$ and all $v \in \mathbb{R}$. Consider the following statements:
 A. The image of f is compact.
 B. The image of f is uncountable.
 Which of the following is correct?
 (1) A is true but B is false.
 (2) B is true but A is false.
 (3) Both A and B are true.
 (4) Both A and B are false.
35. for any pair of subsets C and D of $\mathbb{R}^2$, let C^0 denote the interior of C , and C/D denote the set of points in C which do not belong to D . Consider the following statements:
 A. $\left(\frac{C}{D} \right)^0 = C^0/D^0$
 B. $\left(\frac{C^0}{D} \right)^0 = C^0/D^0$
 Which of the following is necessarily correct?

- (1) A is true but B is false
 (2) B is true but A is false
 (3) Both A and B are true
 (4) Both A and B are false.

36. Consider the subset $S := \{(x, y) \in \mathbb{R}^2 : y = x \text{ or } |x| + |y| = 1\}$ of $\mathbb{R}^2$. Consider the following assertions about S.

- A. S is a reflexive relation on $\mathbb{R}$.
 B. S is a transitive relation on $\mathbb{R}$.
 C. There exists a surjective map from $\mathbb{R}$ onto S.
 D. There exists an injective map from S into $\mathbb{Q}^3$.

Which of the above assertions is/are correct?

- (1) A only (2) A and D only
 (3) A and C only (4) A and B only

37. Let $S = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\} \subset \mathbb{R}$.

Which of the following assertion about S is correct

- (1) For any two maps $f : \mathbb{Q} \times \mathbb{Q} \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow S$, their compositions $g \circ f$ is never surjective
 (2) There exists a surjective map from S onto $\mathbb{R}^2/\mathbb{Z}^2$
 (3) There exists an injective map from $\mathbb{R}/\mathbb{Q}$ into S.
 (4) For any two maps $f : S \rightarrow \mathbb{R}$ and $g : \mathbb{R} \rightarrow \mathbb{R}^3$, their composition $g \circ f$ has a countable range.

38. Consider the following

- A. Union of any collection of open sets in $\mathbb{R}^n$ is open
 B. Intersection of any collection of open sets in $\mathbb{R}^n$ is open. Then
 (1) B is true and A is false.
 (2) A is true and B is false.
 (3) Both A and B are true.
 (4) both A and B are false.

39. Suppose $\mathcal{R}$ Relation defined on $\mathbb{N}_{by}(a, b) \in \mathcal{R}$ - iff $a - b$ is a multiple for 4 or a multiple of 6. Consider the statements:

- A. $\mathcal{R}$ is symmetric.
 B. $\mathcal{R}$ is transitive.
 C. There is an infinite subset $U \subset \mathbb{N}$ or $\mathbb{N}$ such that $\mathcal{R}$ defined on U is an equivalence relation. Then the number of correct statements above is

- (1) 1 (2) 2
 (3) 3 (4) 0

Answer Key

1	2	3	4	5	6	7	8	9	10
4	2	3	2	4	2	3	4	2	2
11	12	13	14	15	16	17	18	19	20
2	3	2	2	1	3	4	3	2	4
21	22	23	24	25	26	27	28	29	30
2	2	4	3	4	2	4	3	4	3
31	32	33	34	35	36	37	38	39	
2	1	3	1	1	4	4	2	2	

EXPLANATIONS

[CUET PG]

1. Explanation:

Let A & B be two non-empty subset of $\mathbb{R}$.

Let

$$A = \mathbb{Q} \quad B = \mathbb{Q}$$

$$\bar{A} = \mathbb{R} \quad A^\circ = \emptyset \quad B' = \mathbb{R}$$

$$\bar{B} = \mathbb{R}$$

$$1. \bar{A} \cup B^\circ = \mathbb{R} \cap \mathbb{Q} = \mathbb{Q}$$

Neither open nor closed (4) incorrect.

$$2. A = [a, b] B = [a, b]$$

$$A^\circ = (a, b) \quad B^\circ = (a, b)$$

$$\bar{A} = [a, b] \quad \bar{B} = [a, b]$$

$$A^\circ \cup B' = [a, b] \text{ - closed set.}$$

So (2) incorrect.

$$A = [0, 2] \quad B = [1, 3]$$

$$A^\circ = (0, 2) \quad \bar{B} = [1, 3]$$

$$B' = [1, 3]$$

$$(1) A^\circ \cap \bar{B} = [1, 2)$$

Option A incorrect.

$$(2) A^\circ \cup B = (0, 3]$$

2 incorrect.

3 correct.

[CU-CET]

1. Explanation:

$$S = \left\{x \in \mathbb{R} : \frac{2x+1}{x+2} < 1\right\}$$

Take $x = 1$

$$\frac{2+1}{3} = 1 < 1 \text{ which not satisfy given condition}$$

$$\text{Take, } x = 0 \quad \frac{1}{2} < 1$$

$$x = -1 \quad \frac{-1}{1} < 1$$

$$x = -3 \quad \frac{-6+1}{-1} = 5 < 1$$

which not satisfy given condition

$$-2 < x < 1$$

$\therefore S$ is bounded both below & above with $\inf s = -2$
 $\sup s = 1$

$\therefore$ option (3) is correct

2. Explanation:

$$S = \left\{ \frac{mn}{m+n+1}, m, n \in \mathbb{N} \right\}$$

$$x = \frac{mn}{m+n+1}$$

$$\Rightarrow x + 1 = \frac{(m+1)(n+1)}{1+m+n}$$

$$= \frac{(m+1)}{m} \frac{(n+1)}{n}$$

$$= \frac{x+1}{x}$$

Now, note that for positive integer k ,

$$\text{we have } 1 < \frac{k+1}{k} \leq 2$$

We can thus infer $\frac{x+1}{x}$ takes value bet. 1 and 4

$$\text{i.e. } 1 < \frac{x+1}{x} \leq 4$$

$$\Rightarrow 0 < \frac{1}{x} \leq 3$$

$$\Rightarrow x \in \left[\frac{1}{3}, \infty \right)$$

We see that $\inf s = 1/3$ & $\sup s = \infty$

$\Rightarrow$ option (2) is correct

3. Explanation:

Let $X = \{x \in \mathbb{R}, x \neq 1, 2, \dots, 100\}$ be subset of $\mathbb{R}$

$$X = \mathbb{R} - \{1, 2, \dots, 100\}$$

$$X^c = \{1, 2, 3, \dots, 100\}$$

$$(X^c)' = \phi$$

X^c is not closed

$\Rightarrow X$ is open

$\Rightarrow$ option (4) is correct

4. Explanation:

$$X = \left\{ \frac{1}{n} \mid n \in \mathbb{Z}, n \neq 0 \right\} \subseteq \mathbb{R}$$

Given set X has upper bound 1 & lower bound 0

$\therefore$ Hence given set is bounded set

$\Rightarrow$ option (1) is correct

5. Explanation:

$(n, n+1)$ is open in $\mathbb{R}$ for any $n \in \mathbb{Z}$

$\therefore U = \bigcup_{n \in \mathbb{Z}} (n, n+1)$ is open in $\mathbb{R}$

(any union of open set is open)

Thus $\mathbb{R} \setminus U$ is closed in $\mathbb{R}$

(a complement of an open set is a closed set)

But $\mathbb{R} \setminus U = \mathbb{Z}$

Thus $\mathbb{Z}$ is closed subset of $\mathbb{R}$

$\Rightarrow$ option (2) is correct

6. Explanation:

$\bigcup_{n=1}^{\infty} (-n, n)$ is subset of $\mathbb{R}$ is connected

$\Rightarrow$ option (3) is correct

7. Explanation:

Every open interval of type (a, b) is homomorphic to $\mathbb{R}$

$\Rightarrow$ option (2) is correct

8. Explanation:

$$U = \left\{ x \in \mathbb{R} : \sin x = \frac{1}{2} \right\}$$

$\sin x : \mathbb{R} \rightarrow [-1, 1]$ is continuous function

Now since $\left\{ \frac{1}{2} \right\}$ is closed in $[-1, 1]$

$$U = \sin^{-1} \left(\frac{1}{2} \right)$$

We get that

U is closed in $\mathbb{R}$

$\Rightarrow$ option (2) is correct

9. Explanation:

Set of boundary point of $\mathbb{Q}^c$ are $\mathbb{Q}^c$

10. Explanation:

$$\text{Let } U = \left\{ x \in \mathbb{R} \mid \sin x = \frac{1}{2} \right\}$$

$$\sin x = \frac{1}{2}$$

$$\Rightarrow x = \frac{\pi}{6} + 2\pi n \quad n \in \mathbb{N} \cup \{0\}$$

Neither open nor closed.

[PRACTICE QUESTIONS]

1. Explanation:

A is closed $A \subseteq \mathbb{R}, A \neq \phi, A \neq \mathbb{R}$

A° – interior of A $\bar{A}$ – closure of A

$$(1) A = \bar{A}^\circ$$

$$A = \{1\} \cup [2, 3] \quad A \subseteq \mathbb{R} \text{ closed } A \neq \phi \quad A \neq \mathbb{R}$$

$$A^\circ = (2, 3) \quad \bar{A}^\circ = [2, 3]$$

$$A = \{1\} \cup [2, 3] \neq [2, 3] = \bar{A}^\circ \Rightarrow A \neq \bar{A}^\circ$$

$$(2) A = [1, 2] \quad A \neq \phi \quad A \neq \mathbb{R} \text{ closed uncountable}$$

$$(3) A = [0, \infty) \quad A \neq \phi \quad A \neq \mathbb{R}$$

Let $B \subseteq \mathbb{R}$ B is closed if B^c is open $A^c = (-\infty, 0) \rightarrow$ open set in $\mathbb{R}$.

$A \neq \text{compact}$.

A is compact in $\mathbb{R}^n$ iff A is closed & bounded ($A \subseteq \mathbb{R}^n$).

(4) $\therefore A$ is not open

Suppose A is open

$\mathbb{R} = A \cup A^c$ $A \subseteq \mathbb{R}$, $A \neq \phi$ A – closed

$A \neq \phi \Rightarrow A^c \neq \mathbb{R}$

$A^c \neq \phi$ $A^c \neq \mathbb{R}$, A^c – open

$A \neq \phi$ $A \neq \mathbb{R}$ A – open

A & A^c forms a separation for $\mathbb{R}$.

$\therefore \Rightarrow \mathbb{R}$ is not connected.

A contradiction

$\Rightarrow A$ not a open

$\Rightarrow$ option (4) is correct.

2. **Explanation:** A and B be two set having m, n many element respectively, where $m < n$, $m, n \in \mathbb{N}$. The number of injective function possible from A to B is

$f : A \rightarrow B$

$|A| = m$ $|B| = n \Rightarrow m < n$

Total injective (one – one) function is nP_m

$\Rightarrow$ option (2) is correct

3. **Explanation:** The set of rational numbers has Archimedean property

$\Rightarrow$ option (3) is correct.

4. **Explanation:**

$\therefore \sqrt{2}$ is an irrational number.

$\therefore \sqrt{2}$ is an algebraic number not transcendent number.

$\therefore$ option (2) is correct.

5. **Explanation:**

$\Rightarrow$ The set of positive primes is countably infinite set $\Rightarrow$ countable

$\Rightarrow$ Set of integer is countable

$\Rightarrow$ The set of rational number is also countable

$\Rightarrow$ The set of irrational number in $[0, 1]$ which is uncountable

$\Rightarrow$ Option (4) is correct

6. Let α be an algebraic number, then by definition, there exist a polynomial, having all coefficient rational, such that $p(\alpha) = 0$. In other words, all roots of a polynomial having all coefficient rational are algebraic number.

Let for a fixed integer n , put

$P_n = \{a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n; a_i \in$

$\mathbb{Q}$ for all $i = 1, 2, 3 \dots n\}$

Then the mapping,

$(a_0 x^n + a_1 x^{n-1} + \dots + a_{n-1} x + a_n) \rightarrow (a_0, a_1, \dots, a_n)$ is P_n to $\mathbb{Q}^{n+1}$

Since, $\mathbb{Q}$ is countable, so is countable product $\mathbb{Q}^{n+1}$.

So set of all polynomials, having rational coefficient is $\bigcup_{n=0}^{\infty} P_n$, which is a countable union of countable sets, and so countable.

Since, a polynomial of degree n , have at most n roots, so the set of all possible roots of all polynomials with rational coefficients is a countable union of finite sets, hence at most countable.

So, the set of all algebraic numbers are countable.

Hence option (2) is correct

Since, $\varepsilon > 0$ was chosen arbitrary. Let $a = \frac{\varepsilon}{2} > 0$ as ε and $\frac{\varepsilon}{2} < \varepsilon$,

7. **Explanation:**

$\therefore$ for all $\varepsilon > 0$, we have $a < \varepsilon$, and $a \neq 0$.

Hence option (3) is correct

8. **Explanation:** Statement in option (D) is incorrect, as $\bar{T}$ is the smallest closed set which contain T

For example let $T = (0, 1)$ is set in $\mathbb{R}$ and here $\bar{T} = [0, 1]$ and $T \subset \bar{T}$

But this is not the largest closed set as $S = [10, 8]$ is also closed set containing T .

(we take $[0, 8]$ as an example)

Hence option (4) is correct.

$$a_n = \begin{cases} 2^n + n^2, & n \leq 100 \\ 3 + (-1)^n \frac{n^2}{2^{n+1}}, & n > 100 \end{cases}$$

Note that to check the convergence or divergence of a sequence, if we remove some finite number of terms, it won't have any effect on the convergence or divergence.

For large n , $n^2 < 2^n + 1$, and $\lim_{n \rightarrow \infty} \frac{n^2}{2^{n+1}} \left(\frac{\infty}{\infty} \text{ form} \right)$

Applying L'Hopital Rule

$$\lim_{n \rightarrow \infty} \frac{n^2}{(\log 2) \cdot 2^n} \left(\frac{\infty}{\infty} \text{ form} \right)$$

$$\lim_{n \rightarrow \infty} \frac{n^2}{2^n \cdot (\log 2)^2} \left(\frac{\infty}{\infty} \text{ form} \right)$$

$$\lim_{n \rightarrow \infty} a_n = 3$$

i.e., $\langle a_n \rangle$ is convergent sequence, hence Cauchy.

Hence option (1) is correct

9. **Explanation:**

For option (A): Note that for all $n \in \mathbb{N}$, $\frac{1}{n} > 0$

Therefore, $0 \notin \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right]$

$\Rightarrow 0 \notin \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right]$

$\bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 2\right] \neq [0, 2]$

For option (B): Let $x \in (0, 2]$ be arbitrary, then $0 < x$, then there exist a natural number n such that $0 < \frac{1}{n} < x$,

for all $x > 0$, we have $n \in \mathbb{N}$ such that $(x, 2] \subset \left(\frac{1}{n}, 2\right]$

Since $x > 0$ was arbitrary

we have $(0, 2] \subseteq \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right]$ (1)

If $y \in \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right]$ is arbitrary, then $y \in \left(\frac{1}{n}, 2\right]$ for some $n \in \mathbb{N}$

i.e. $y > \frac{1}{n} > 0$

$\therefore \bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right] \subseteq (0, 2]$ (2)

From (1) and (2),

we have $\bigcup_{n=1}^{\infty} \left(\frac{1}{n}, 2\right] = (0, 2]$

For option (C) Note that for all $n \in \mathbb{N}$, $1 - \frac{1}{n} < 1$,

Also, $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1$

Therefore, $1 \in \bigcup_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right]$, But $1 \notin (1, 2]$

Hence, $\bigcup_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right]$ is arbitrary

For Option (D) Let $x \in [1, 2]$ be arbitrary then $1 \leq x \leq 2$

. Since $\forall n \in \mathbb{N}$, $1 - \frac{1}{n} < 1 \leq x$,

$\Rightarrow x \in \left[1 - \frac{1}{n}, 2\right] \forall n \in \mathbb{N}$

Since, x was arbitrary, therefore $[1, 2] \subseteq \bigcup_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right]$ (3)

Let $y \in \bigcup_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right]$ is arbitrary

Then, $y \in \left[1 - \frac{1}{n}, 2\right] \forall n \in \mathbb{N}$

Also, $\lim_{n \rightarrow \infty} \left(1 - \frac{1}{n}\right) = 1$, then $y \geq 1$.

So, $y \in [1, 2]$

Since y was choose arbitrary, there fore $\bigcup_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right] \subseteq [1, 2]$ (4)

From (3) and (4),

We have, $\bigcap_{n=1}^{\infty} \left[1 - \frac{1}{n}, 2\right] = [1, 2]$

Hence option (2) and (4) are correct.

10. Explanation: For option (A) and (C) Let us consider a set

$$A = (-\infty, \sqrt{2}) \cup (\sqrt{2}, \infty)$$

Clearly, A is open, as union of two open intervals

Also, note that $A \neq \mathbb{R}$ as $\sqrt{2} \in \mathbb{R}$ But as $\sqrt{2} \notin A$ and is uncountable set.

Therefore, $Q \subset A \subset \mathbb{R}$, But $A \neq \mathbb{R}$ and A is uncountable but not closed.

It contradicts both (A) and (C)

Also, option (D) is wrong, as if we consider $A = \mathbb{Q} \cup \{\sqrt{2}, x\}$ clearly $Q \subset A \subset \mathbb{R}$. Also, A is countable, as union of two countable set. But A is not closed in $\mathbb{R}$

Also, option (2) is correct, as given that $Q \subset A$ then $\overline{Q} \subset \overline{A}$ Then $\overline{Q} \subseteq \overline{A}$. But $\overline{Q} = \mathbb{R}$ and $\overline{A} = A$ (As A is closed)

Then we have $\mathbb{R} \subseteq A$, this given $A = \mathbb{R}$.

Hence option (2) is correct

11. Explanation:

For option (A) Since $\mathbb{N}'$ is the countable product of countable sets. Since countable product of countable sets is countable. So, $\mathbb{N}'$ is countable

(C) the set of integers is countable

(D) Since, $\mathbb{Q}$ is countable and $\sqrt{2} \in \mathbb{Q}$ then $\sqrt{2}\mathbb{Q}$ is also countable.

(B) $\{0, 1\}^{\mathbb{N}}$, is the set of all sequences which takes the value either 0 and 1.

Suppose that this set is countable then there exist bijection with $\mathbb{N}$.

Let, $\{0, 1\}^{\mathbb{N}} = \{x_{1n}, x_{2n}, x_{3n} \dots\}$, where each x_{in} , is a sequence and $f(i) = x_{in}$

Let us consider a sequence X , such that

$X_1 \neq X_{11}$ (if $X_{11} = 0$, we define $X_1 = 1$ or vice versa)

$X_2 \neq X_{22}$ (same as above)

:

:

$X_n \neq X_{nn}$

Hence, $\langle x_n \rangle$ is a sequence having value either 0 or 1 but not in the range set of f . Hence $\{0, 1\}^{\mathbb{N}}$ is Uncountable
Hence, option (2) is correct.

12. Explanation:

For example, Consider, $X = \mathbb{Z}$ and $A = \mathbb{N}$

Clearly, both are countably infinite subset of $\mathbb{R}$, then

$$X \setminus A = \{x \in X | x \notin A\}$$

$$= \{x \in \mathbb{Z} | x \notin \mathbb{N}\}$$

$$= \{0, -1, -2, \dots\} \text{ is countable infinite set.}$$

Also $X \setminus A \subset X$, then X cannot be uncountable.

13. Explanation: consider the finite series

$$\sum u_n = x + \frac{2^2 x^2}{|2|} + \frac{3^3 x^3}{|3|} + \dots$$

$$\text{Hence, } u_n = \frac{n^n x^n}{|3|} \text{ and } u_{n+1} = \frac{(n+1)^{n+1} x^{n+1}}{|n+1|}$$

$$\therefore \frac{u_{n+1}}{u_n} = \frac{(n+1)^{n+1} \cdot x^{n+1} \cdot |n|}{|n+1| \cdot n^n \cdot x^n} = \frac{(n+1)^n}{n^n} \cdot x$$

$$= \left(1 + \frac{1}{n}\right)^n \cdot x$$

$$\text{Therefore, } \lim_{n \rightarrow \infty} \frac{u_{n+1}}{u_n} = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^n \cdot x = ex$$

By D'Alembert ratio test given infinite series is convergent $ex < 1$ i.e. $x < \frac{1}{e}$

Hence option (1) is correct

(Note that because of options, we didn't check the convergence for $ex = 1$, otherwise we need to check.)

14. Explanation:

$$A = \{x \in \mathbb{Q} : -1 < x < 0\} \cup \mathbb{N}$$

$$= ((-1, 0) \cap \mathbb{Q}) \cup \mathbb{N}$$

Clearly, A is unbounded set, as $\mathbb{N}$ is unbounded.

Clearly, A is infinite set as both natural number and rational between -1 and 0, are infinite in numbers.

Clearly, $\mathbb{N}$ have no limit point, But the set of limit points of $(-1, \mathbb{Q}) \cap \mathbb{Q}$ is the whole interval $[-1, 0]$ in $\mathbb{R}$.

Hence option (2) is correct.

15. Explanation: Since, $S = \prod_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$

$$= \prod_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$$

Let $x \in [0, 1]$ is arbitrary, then for all $n \in \mathbb{N}$,

$$\text{we have } -\frac{1}{n} < 0 \leq x \leq 1 < 1 + \frac{1}{n}$$

$$\Rightarrow x \in \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] \text{ for all } n \in \mathbb{N}$$

Since x was arbitrary. Therefore

$$[0, 1] \subseteq \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$$

$$\text{Now let } y \in \bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right]$$

$$\Rightarrow y \in \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] \text{ for all } n \in \mathbb{N}$$

$$\Rightarrow -\frac{1}{n} \leq y \leq 1 + \frac{1}{n} \forall n \in \mathbb{N}$$

$$\text{Since, } \lim_{n \rightarrow \infty} \left(-\frac{1}{n} = 0\right) = 0 \text{ and } \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right) = 1 + 0 = 1$$

Then by squeeze, theorem

$$0 \leq y \leq 1 \text{ i.e. } y \in [0, 1]$$

Since y was arbitrary, we have

$$\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] \subseteq [0, 1]$$

From (1) and (2), we have

$$\bigcap_{n=1}^{\infty} \left[-\frac{1}{n}, 1 + \frac{1}{n}\right] = [0, 1]$$

Hence option (1) is correct

16. Explanation: Given that

$$S = \left\{\frac{1}{n} : n \in \mathbb{N} \text{ and } n \text{ is prime}\right\}$$

$$= \left\{\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \dots\right\}$$

Carty, $p \geq 2$ for every prime $p \in \mathbb{N}$.

$$\therefore \frac{1}{p} \leq \frac{1}{2} \text{ for every prime } p \in \mathbb{N}.$$

$$\Rightarrow \sup(S) = \frac{1}{2}$$

(as $\frac{1}{2}$ is an upper bound of S and element of S)

$$\text{Also, } T = \{x^2 : x \in \mathbb{R}\}$$

$$\text{Clearly, } x^2 \geq 0 \forall x \in \mathbb{R}$$

Hence, is a lower bound of T and $0 \in T$ (as $x = 0 \in \mathbb{R}$), therefore $\inf T = 0$

Here option (3) is correct.

17. Explanation:

(I) Let, us define $f: \mathbb{N} \rightarrow I$ as $f(n) = \frac{1}{n}$ clearly f is one-one correspondence.

(II) $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ is the set of all integers and integers are countable hence there must exist a one-one correspondence with natural number.

(III) $\left\{\frac{p}{q} : p, q \in \mathbb{N}\right\}$ is the set of all Rational numbers, and set of rational number is countable hence there must exist a one-one correspondence with natural number.

(IV) Clearly, $\left\{\frac{p}{q} : p, q \in \mathbb{N}\right\}$ is a subset of $\mathbb{Q}$

Since, $\mathbb{Q}$ is countable, so being a subset of countable set, (IV) is also countable.

Since it is infinite set, so there must exist a one-one correspondence with natural number.

Hence option (4) is correct.

18. Explanation: $X = \{(x, y) \in \mathbb{R}^2 : x \in \mathbb{Q}, y \in \mathbb{R} \setminus \mathbb{Q}\}$

Now we have to check whether X is open in $\mathbb{R}^2$ or dense in $\mathbb{R}^2$.

Open: $(\alpha, \beta) \in X$ be arbitrary point, then every neighbourhood of (α, β) in $\mathbb{R}^2$ contain a point where $\alpha \notin \mathbb{Q}$ and $\beta \notin \mathbb{Q}^c$ so (α, β) need not be interior point of X , so X is not open

Dense: Let $(a, b) \in \mathbb{R}^2$ Then every neighbourhood of (a, b) in $\mathbb{R}^2$ contains point where $a \in \mathbb{Q}, b \in \mathbb{Q}^c$ and $a \in \mathbb{Q}^c, b \in \mathbb{Q}$

i.e. neighbourhood of $(a, b) \in X$ and $(a, b) \in X^c$

i.e. (a, b) is a boundary point of $\mathbb{R}^2$