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CHAPTER 1

REAL ANALYSIS

Unit 1.1: Set Theory and Countability

[Previous Year Exam Questions]

[MH-2023]

1. Let Q be the set of rational numbers and $\mathbb{R}$ be the set of real numbers.
   Then which one of the following statements is true?
   (a) Q is not an ordered field
   (b) Q is a complete ordered field
   (c) $\mathbb{R}$ is not a complete ordered field
   (d) $\mathbb{R}$ is a complete ordered field

[MH-2023]

2. Which of the following pair X, Y of sets do not have same cardinality, i.e. $|X| = |Y|$?
   (a) $X = [0,1], Y = [0,1] \times [0,1]$
   (b) $X =$ the set of irrationals,
   $Y =$ the set of algebraic numbers
   (c) $X = \mathbb{R}, Y = C$
   (d) $X =$ a bounded interval $[x, y]$ with $x < y$ in $\mathbb{R}, Y = (-\infty, \infty) = \mathbb{R}$

Answer Key	
1	**2**
(d)	(b)

EXPLANATIONS

1. Explanation:
$\mathbb{R}$ is complete order field.

2. Explanation:
$x = Q^c$
Y= set of algebraic number
$|x| \neq |Y|$

Unit 1.2: Point Set Topology

[Previous Year Exam Questions]

[MH-AUG-2011]

1. The segment (a, b) of the x-axis of $\mathbb{R}^2$ is:
   (a) Neither open nor closed
   (b) Open
   (c) Closed
   (d) Both open and closed

[MH-AUG-2011]

2. An open set in $\mathbb{R}$ is:
   (a) an interval
   (b) finite union of open intervals
   (c) countable union of disjoint open intervals in $\mathbb{R}$
   (d) intersection of open intervals

[MH-AUG-2011]

3. If $A_n = \left[\frac{1}{n}, 1\right]$, then $\bigcup_{n=1}^{\infty} A_n$ is:
   (a) (0,1) (b) (0,1]
   (c) [0,1] (d) [0,1)

[MH-AUG-2011]

4. If a and b are real numbers, then $\sup\{a, b\} =$
   (a) $\frac{a+b-|a-b|}{2}$ (b) $\frac{a+b+|a-b|}{2}$
   (c) $\frac{a-b+|a+b|}{2}$ (d) $\frac{a-b-|a-b|}{2}$

[MH-AUG-2011]

5. If $I_n = \left\{x \epsilon \mathbb{R} \middle| 0 < x < \frac{1}{n}\right\}$, then $\bigcap_{n=1}^{\infty} I_n$ is:
   (a) $\{0\}$ (b) ϕ
   (c) $\{x \epsilon \mathbb{R} | -1 < x < 1\}$ (d) $\mathbb{R}$

[MH-DEC-2013]

6. Let $G_n = \left(0, 1 + \frac{1}{n}\right)$ for $n \in$ N. Then $\cap G_n$ is :
   (a) closed
   (b) open
   (c) both open and closed
   (d) neither open nor closed

[MH-DEC-2013]

7. Let $A_n = \left[\frac{1}{n}, 1\right]$. Then $\bigcup_{n=1}^{\infty} A_n$ is :
   (a) (0, 1) (b) (0, 1]
   (c) [0, 1] (d) [0, 1)

[MH-DEC-2013]

8. In R let $F_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \forall\ n \in N$. Then, $\cap F_n$ is :
   (a) [0]
   (b) ϕ
   (c) both open and closed
   (d) neither open nor closed

[MH-FEB-2013]

9. Which of the following is true?
   A subset of A of $\mathbb{R}^n$ is compact if
   (a) the complement of A is an open set
   (b) A is connected and closed
   (c) A is closed and bounded
   (d) every limit point of A is in A

[MH-FEB-2013]

10. Which of the following real valued functions defined on [0, 1] is not dense in C[0, 1]?
    (a) The set of Bernstein polynomials
    (b) The set of polygonal functions with rational values at nodes $\left\{\frac{0}{1}, \frac{1}{n}, \ldots \frac{n}{n}\right\}, n = 2,3,4 \ldots \ldots$
    (c) The set of all polynomials
    (d) The set of all polynomials of degree ≤ 10

[MH-2019]

11. Limsup of the sequence $\left\{-2,2,-\frac{3}{2},\frac{3}{2},-\frac{4}{3},\frac{4}{3} \ldots \ldots .\right\}$ is:
    (a) 3/2 (b) 2
    (c) 1 (d) 0

[MH-2019]

12. Let X and Y be bounded subsets of $\mathbb{R}$ and let $Z = X \cup Y$. Then:
    (a) Z need not be bounded.
    (b) Sup Z = max {sup X, sup Y}
    (c) Z need not have supremum
    (d) Z has maximum

[MH-2023]

13. Suppose that $\{A_n\}$ is a sequence of sets such that:
    $A_n = \begin{cases} (n, n+1, n+2) & if\ n\ is\ even \\ (n, n+1, n+2, \ldots .) & if\ n\ is\ odd \end{cases}$
    Then, which of the following is/are true?
    (a) $\lim_{n\to\infty} A_n$ does not exist
    (b) $\lim_{n\to\infty} A_n = \{1,2,3, \ldots . .\}$
    (c) $\lim_{n\to\infty} A_n = \{3,5,7, \ldots\}$
    (d) $\lim_{n\to\infty} A_n = \phi$

[MH-2023]

14. Consider the set given by $X = \left(2m + \frac{1}{m} \in \mathbb{Z}\right)$ and $Y = \left(2n + \frac{3}{n} n \in \mathbb{Z}\right)$
    Then:
    (a) X is a proper subset of Y
    (b) Y is a proper subset of X

(c) $X = Y$

(d) Either X is finite or Y is finite

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
(b)	(c)	(b)	(b)	(b)	(b)	(b)	(b)	(c)	(d)
11	**12**	**13**	**14**						
(b)	(d)	(d)	(c)						

EXPLANATIONS

1. Explanation:

The segment (a, b) of the x-axis of $\mathbb{R}^2$

(a, b) is not closed since it does not contain limit point

Since $\mathbb{R}$ is closed as it's complement is open

So (a, b) is open

2. Explanation:

Every open subset U of R can be uniquely expressed as a countable union of disjoint open Interval in R.

3. Explanation:

If $A_n = \left[\frac{1}{n}, 1\right]$

$A_1 = [1,1]$,

$A_2 = \left[\frac{1}{2}, 1\right]$

$A_3 = \left[\frac{1}{3}, 1\right]$

$A_4 = \left[\frac{1}{4}, 1\right]$

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$\bigcup_{n=1}^{\infty} A_n = [1,1] \cup \left[\frac{1}{2}, 1\right] \cup \left[\frac{1}{3}, 1\right]$

$\bigcup_{n=1}^{\infty} A_n = (0,1]$

4. Explanation:

If a & b are real numbers

Let $a = 2\ b = 3, \sup\{2,3\} = 3$

(a) $\frac{a+b-|a-b|}{2} = \frac{5-|-1|}{2} = \frac{4}{2} = 2$

(b) $\frac{a+b+|a-b|}{2} = \frac{5+|-1|}{2} = \frac{6}{2} = 3$

(c) $\frac{a-b+|a+b|}{2} = \frac{2-3+|5|}{2} = \frac{-1-1}{2} = -1$

(d) $\frac{a-b-|a-b|}{2} = \frac{2-3-|2-3|}{2} = \frac{-1-1}{2} = -1$

This is open set

From this we can see (a) (c) & (d) is incorrect.

5. Explanation:

$I_n = \left\{x \in \mathbb{R} \middle| 0 < x < \frac{1}{n}\right\}$

$I_1 = \{x \in \mathbb{R} | 0 < x < 1\} = (0,1)$

$I_2 = \left\{x \in \mathbb{R} \middle| 0 < x < {}^1/_2\right\} = \left(0, {}^1/_2\right)$

$I_3 = \left\{x \in \mathbb{R} \middle| 0 < x < {}^1/_3\right\} = \left(0, {}^1/_3\right)$

$\bigcap_{n=1}^{D} I_n \quad = (0,1)\, n \left(0, {}^1/_2\right) n \left(0, {}^1/_3\right) \ldots . n \{0 =$

$\{\} = \phi$ '

6. Explanation:

$G_n = \left(0,1 + \frac{1}{n}\right) = (0,1)$

Open set

7. Explanation:

$\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} \left[\frac{1}{n}, 1\right] = (0,1]$

8. Explanation:

$F_n = \left(-\frac{1}{n}, \frac{1}{n}\right), \forall n \in \mathbb{N}$

$F_1 = (-1,1)$

$F_2 = \left(\frac{-1}{2}, \frac{1}{2}\right)$

$F_3 = \left(-{}^1/_3, {}^1/_3\right)$

.

.

.

$\cap F_n = (-1,1) \cap \left(\frac{-1}{2}, \frac{1}{2}\right)$

$\lim_{n \to S} \left(-{}^1/_n, {}^1/_n\right) = (0,0) = \phi$

$\cap F_n = \phi.$

9. Explanation:

A subset of A of $\mathbb{R}^n$ is compact if A is closed and bounded

10. Explanation:

Here $I_n = [a_n, b_n]$

$\inf\ \{b_n - a_n\} = 0$

As I_n is decreasing

$\Rightarrow I_{n+1} C\ I_n$

Let $\left[\frac{-1}{n}, \frac{1}{n}\right] = I_n$

$\cap I_n = \{0\}$

$\Rightarrow I_n$ has exactly one point

11. Explanation:

$a_n \left\{-2,2,\frac{-3}{2},\frac{3}{2},\frac{-4}{3},\frac{4}{3}\ldots\right\}$

$\sup a_n = 2$

12. Explanation:

$Z = X \cup Y$

$X, Y, \subseteq \mathbb{R}$

Let X, Y have maximum

Let x_1, x_2 be maximum $oy\ x \cup y$

$\Rightarrow X \cup Y$

13. Explanation:

$A_n = \begin{cases}(n, n+1, n+2, \ldots) \\ (n, n+1, n+2 \ldots)\end{cases}$

$A_n = \begin{cases}B_n & n\ is\ even \\ C_n & n\ is\ odd\end{cases}$

$C_n = (n_1, n+1, n+2, \ldots)$

$c_{n+1} = (n+1, n+2, \ldots)$

$C_{n+2} = (n+2\ n+5, , \ldots.)$

$C_{n+1} \subseteq C_{n+1} \subseteq C_n$

$\lim A_n = \lim A_n = \phi$

14. Explanation:

$x = \left\{2m + \frac{1}{m} \in \mathbb{Z}\right\}$

$Y = \{2n + 3/n \in \mathbb{Z}\}$

Both are odd Integer's

$\Rightarrow X = Y$