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CHAPTER 1

SET THEORY & COUNTABILITY

SET THEORY

1.1 SETS :

1.1.1 DEFINITION:

A Set is a well-defined collection of distinct objects.

The individual objects of the set are called member or elements of the set.

Set is denoted by Capital letters A, B, C, X, Y, Z ...

Examples :-

(1) Set of all vowels in English Alphabet $X = \{a, e, i, o, u\}$.

(2) Some Important sets : $\mathbb{N}$ = set of all natural numbers

$\mathbb{Z}$ = set of all integers

$\mathbb{R}$ = set of all real numbers

$\mathbb{C}$ = set of all complex numbers

$\mathbb{Q}$ = set of all rational numbers

$\mathbb{Q}^c$ = set of all irrational numbers

(3) The set of all natural numbers which are strictly less than 4.

$X = \{x \in \mathbb{N} \mid x < 4\} = \{1,2,3\}$

Note:

(1) Set guarantees that the Collection will contains distinct elements but collection doesn't guarantee this.

(2) The word 'Well-defined' means there is no ambiguity or confusion as to whether an object belongs to collection or not.

1.1.2 Equality of Two sets :

Two sets are said to be equal when they consist of exactly the same elements.

(i) $\{a, b, c\} = \{b, c, a\} = \{c, b, a\}$

(ii) $\{x \in \mathbb{N} \mid x < 4\} = \{x \in \mathbb{N} | x \leq 3\} = \{1,2,3\}$

Cardinality of the set :

The number of elements in the set A is called cardinality of the set A

Cardinality of the set A is denoted by $|A|$.

Russel's Paradox :

Let R be the collection of all sets that are not members of themselves.

$R = \{A | A \notin A\}$

Exercise: Show that $R \notin R \Leftrightarrow R \in R$.

1.1.3 Subset :

If A & B are two sets such that each member of A is also a member of B

i.e. $x \in A \Rightarrow x \in B$

Then A is called subset of B.

OR,

$\nexists\ x \in A$ such that $x \notin B$.

Notation:

(1) If A is subset of B, then it is denoted by $A \subseteq B$(B is superset of A)

(2) If A is proper subset of B, then it is denoted by $A \subsetneq B$ (i.e. $\exists\ x \in B$ such that $x \notin A$).

Example :

1) If $x \in A \Rightarrow x \in A$

$\Rightarrow A$ is subset of A i.e $A \subseteq A$

$\therefore$ Every set is a subset of itself.

2) ϕ (empty set) is a subset of X i.e. $\phi \subseteq X$.

1.1.4 Power Set :

The collection of all subsets of a set A is called Power set of A. It is denoted by $P(A)$.

Example :

(1) $A = \{1,2,3\}$

$P(A) = \{\phi, A, \{1\}, \{2\}, \{3\}, \{1,2\}, \{1,3\}, \{2,3\}\}$.

Note:- for any set A

(1) $\phi \in P(A)$

(2) ϕ is element as well as subset of $P(A)$.

1.1.5 Cartesian Product :

Let A & B are two sets. Then Cartesian product of A & B is denoted by $A \times B$ and it is defined as

$A \times B = \{(a, b) | a \in A, b \in B\}$, where (a, b) is an order pair.

Example:-

(1) $A = \{1,2\}, B = \{3,4\}$

$A \times B = \{(1,3), (1,4), (2,3), (2,4)\}$

(2) $A = \{1,2\}, B = \phi$, then

$A \times B = \phi$

1.1.6 Union & Intersection of two sets :

(1) Union :

If A & B are two sets, then the set consisting of all elements which belongs to either A or B is defined as union of A & B.

It is denoted by $A \cup B$

$A \cup B = \{x \mid x \in A \text{ or } x \in B\}$

(2) Intersection :

If A & B are two sets, then the set consisting of all elements which belongs to both A & B is defined as intersection of A & B

It is denoted by $A \cap B$

$A \cap B = \{x | \, x \in A \ and \ x \in B\}$

Note:

(1) Two Set A & B are said to be disjoint if they have no common elements.

i.e. $A \cap B = \phi$

(2) $A \cup \phi = A$, $A \cup A = A$, $A \cup B = B \cup A$

$A \cap \phi = \phi$, $A \cap A = A$, $A \cap B = B \cap A$.

1.1.7 Union & Intersection of an arbitrary family of sets :

Index Set :

An index set is a set whose members labels (or index) members of another set. It is denoted by Λ

Arbitrary Union & Intersection

Let Λ be an index set, then

$\bigcup_{i \in \Lambda} A_n = \{x | \, x \in A_i, \text{ for some } i \in \Lambda\}$

and

$\bigcap_{i \in \Lambda} A_n = \{x | \, x \in A_i, \text{ for each } i \in \Lambda\}$

1.2 Relations :

1.2.1 Definition:

If A & B be two non-empty sets, then a relation from A to B is a subset of $A \times B$ & every subset of $A \times B$ is a relation from A to B

A relation from A to A is called binary relation.

Examples :

1) Null relation $\phi \subseteq A \times B$

2) $A = \{1, 2\}$, & $B = \{0, 1, 2\}$, then

$R_1 = \{(1, 1), (2, 2)\}$ is a relation.

$R_2 = \{(1, 0), (1, 2), (2, 1), (2, 2)\}$ is a relation

$R_3 = \{(1, 0), (0, 1)\}$ is not a relation.

1) Reflexive Relation :

Let R be a binary relation on a set A i.e. $R \subseteq A \times A$.

Then R is called a reflexive relation if

$(a, a) \in R, \forall \, a \in A$

Example :

$A_1 = \{1, 2, 3\}$ & $R_1 = \{(1, 1), (2, 2), (3, 3), (1, 2), (2, 3)\}$

$A_2 = \{a, b\}$ & $R_2 = \{(a, a), (b, b), (a, b), (b, a)\}$

Both are reflexive relation.

2) Symmetric Relation:

Let R be a binary relation on a set A i.e. $R \subseteq A \times A$.

Then R is called a symmetric relation if

$(a, b) \in R \Rightarrow (b, a) \in R$

Example :

$A = \{1, 2, 3\}$ & $R = \{(1, 1), (1, 2), (2, 1), (1, 3), (3, 1)\}$ is symmetric relation.

3) Transitive Relation :

Let R be a binary relation on a set A i.e. $R \subseteq A \times A$.

Then R is called a transitive relation if

(a, b) & $(b, c) \in R$ then $(a, c) \in R$

Example :

$A_1 = \{1, 2, 3\}$, $R_1 = \{(1, 1), (1, 2), (2, 3), (1, 3)\}$ is transitive relation.

Equivalence Relation :

A binary relation $R \subseteq A \times A$ is called equivalence relation if it is reflexive, symmetric & transitive.

1.2.2 Equivalence class :

If $R \subseteq A \times A$ is an equivalence relation. Then equivalence class of A with respect to R is denoted by $[a]$ & is defined as

$[a] = \{x \in A | \, (a, x) \in R\}$

Result :-

(1) Two equivalence class are either equal or disjoint.

(2) Fundamental Theorems of equivalence relation.
An equivalence relation on a set A form a partition of A conversely, Corresponding to any partition of A, there exist an equivalence relation on A.

(3) The number of equivalence relation on a set A is equals to the number of partition of a set A.

1.2.3 Number of Equivalence relations :

Let A be any set with cardinality $|A|$, then

$\|A\|$	Number of equivalence relation
0	1
1	1
2	2
3	5
4	15
5	52

Definition :

Let $S = \phi$, & $R \subseteq S \times S$ be a relation on S, then R is null relation & it is an equivalence relation.

Anti-symmetric Relation:

A binary relation R is anti-symmetric relation if

$(a, b) \in R$ & $(b, a) \in R \Rightarrow a = b$

1.2.4 Partial order Relation :

A relation R on a set A is said to be partial order relation if it is

(a) Reflexive

(b) Anti symmetric

(c) Transitive

Example:

Let $x, y \in \mathbb{R}$, define xRy iff $x - y + \sqrt{2} \in \mathbb{Q}^c$. Then R is

(a) Reflexive relation

(b) Symmetric relation

(c) Transitive relation

(d) Equivalence relation

(e) Anti-symmetric relation

(f) Partial order relation

Solution:

(a) If $y = x$ then $x - y + \sqrt{2} = x - x + \sqrt{2} = \sqrt{2} \in \mathbb{Q}^c$

$\Rightarrow x - y + \sqrt{2} \in \mathbb{Q}^c$

$\Rightarrow x\ R\ x$

$\Rightarrow R$ is a reflexive relation.

(b) $x = 1 + \sqrt{2}, y = 1$

$x - y + \sqrt{2} = 1 + \sqrt{2} - 1 + \sqrt{2} = 2\sqrt{2} \in \mathbb{Q}^c$

$\Rightarrow xRy$

Consider $y - x + \sqrt{2} = 1 - (1 + \sqrt{2}) + \sqrt{2} = 1 - 1 - \sqrt{2} + \sqrt{2} = 0 \notin \mathbb{Q}^c$

$\Rightarrow y$ is not related to x

$\Rightarrow R$ is not a symmetric relation.

(c) $x = \frac{-\sqrt{2}}{2}, y = 1, z = \frac{\sqrt{2}}{2}$

$x - y + \sqrt{2} = \frac{-\sqrt{2}}{2} - 1 + \sqrt{2} = \frac{-\sqrt{2} - 2 + 2\sqrt{2}}{2} = \frac{-2 + \sqrt{2}}{2} \in \mathbb{Q}^c$

$y - z + \sqrt{2} = 1 - \frac{\sqrt{2}}{2} + \sqrt{2} = \frac{2 - \sqrt{2} + 2\sqrt{2}}{2} = \frac{2 + \sqrt{2}}{2} \in \mathbb{Q}^c$

$\Rightarrow xRy\ \&\ yRz$

But $x - z + \sqrt{2} = \frac{-\sqrt{2}}{2} - \frac{\sqrt{2}}{2} + \sqrt{2} = -\frac{2\sqrt{2}}{2} + \sqrt{2} = -\sqrt{2} + \sqrt{2} = 0 \notin \mathbb{Q}^c$

$\Rightarrow R$ is not a transitive relation

(d) Clearly R is not an equivalence relation

(e) $x = \sqrt{2}, \quad y = \sqrt{3}$

$x - y + \sqrt{2} = \sqrt{2} - \sqrt{3} + \sqrt{2} = 2\sqrt{2} - \sqrt{3} \in \mathbb{Q}^c$

$\Rightarrow xRy$

$y - x + \sqrt{2} = \sqrt{3} - \sqrt{2} + \sqrt{2} = \sqrt{3} \in \mathbb{Q}^c$

$\Rightarrow yRx$

But $x \neq y$

$\Rightarrow R$ is not an anti-symmetric relation

(f) Clearly, R is not a partial order relation.

EXERCISE

Let $x, y \in \mathbb{Z}$. Define xRy iff $x = y \pmod 3$. Then R is

(a) Reflexive relation

(b) Symmetric relation

(c) Transitive relation

(d) Equivalence relation

(e) Anti-symmetric relation

(f) Partial order relation

(If R is an equivalence relation, find the equivalence classes)

1.3 Function:

1.3.1 Definition:

Let A & B be any two non-empty sets. Then a rule which assign each member x of A to unique member y of B is called a function and y is the image of x under f.

Notation: $f : A \to B$ i.e. f is a mapping or a function from A to B.

Note :

(i) A function is a special type of relation in which each element of A is pair with exactly one element of B.

i.e. if $(a, b) \in f \Rightarrow f(a) = b$,

1.3.2 Definition:

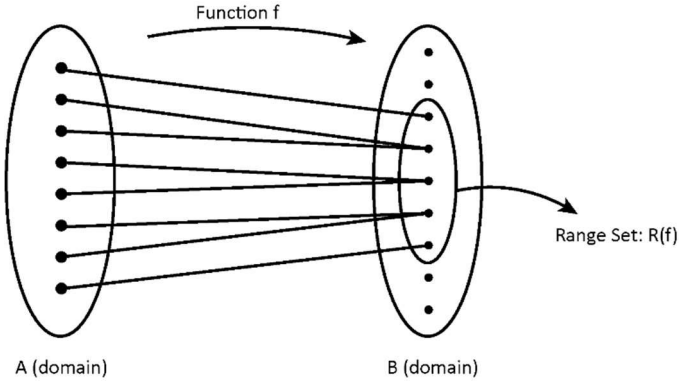
Domain, Co-domain & Range of :

Let $f : A \to B$ be any function from A to B then

i) The set A is called the domain of the function f.

ii) The set B is called the co-domain of the function f.

ii) The set of all elements in B which are the image of the elements of the domain A is called the range set of the function f range of $f = R(f) = \{y \in B \mid y = f(x) \text{ for some } x \in A\}$



1.3.3 Counting Number of functions from A to B :

Let $|A| = m$ & $|B| = n$, then the number of functions from A to B are $|B|^{|A|} = n^m$

Example :

Q. If $|A| = 3$ & $|B| = 5$, then find number of functions from A to B.

Solution :

If $|A| = m, |B| = n$ &

$F = \{f \mid f : A \to B\}$ then

$|F| = n^m$

Here $m = 3$ and $n = 5$

$\therefore |F| = n^m = 5^3 = 125$

Q. Let $|A| = m$ & $|B| = n$ defline
$F_1 = \{f | f : A \to B$ is a function$\}$ &
$F_2 = \{f | f : B \to A$ is a function$\}$. Then total number of functions form F_2 to F_1 are

(a) m^{n^m} (b) $n^{m^{n+1}}$
(c) $m^{n^{m+1}}$ (d) n^{m^n}

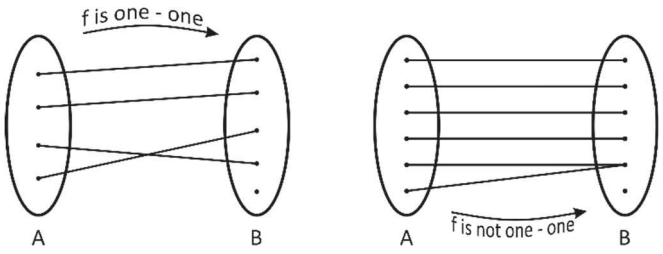
Solution: Here $|A| = m, |B| = n$
$\therefore |F_1| = |B|^{|A|} = n^m$
$|F_2| = |A|^{|B|} = m^n$
$F = \{f | f : F_2 \to F_1 \; be \; a \; function\}$
Then
$|F| = |F_1|^{|F_2|} = (n^m)^{(m^n)}$
$= n^{m \times m^n} = n^{m^{n+1}}$
$|F| = n^{m^{n+1}}$
$\therefore$ Number of functions from F_2 to F_1 are $n^{m^{n+1}}$

1.4 Classification of function:-

On the basis of mapping of elements, there are five types of functions
(1) one to one function (Injective function)
(2) Many to one function
(3) Onto function (Surjective function)
(4) Into function
(5) One-one & onto function (Bijective function or Invertible function)

1.4.1 One-one function (Injective function):

A function $f : A \to B$ is said to be one-one function, if two distinct elements of the domain have distinct images in the co-domain.
i.e. f is one-one iff for any $x_1, x_2 \in A$ if $x_1 \neq x_2 \Rightarrow f(x_1) \neq f(x_2)$
OR
If $f(x_1) = f(x_2) \Rightarrow x_1 = x_2$



Counting number of one-one function from A to B:
If $|A| = m$ & $|B| = n$. then total number of one-one functions from A to B are
$= \begin{cases} {}^n p_m & m \leq n \\ 0 & m > n \end{cases}$ where ${}^n p_m = \frac{n!}{(n-m)!}$

Example :-
Q. Let $A = \{1, 2, 3\}$ & $B = \{1, 2, 3, 4, 5\}$
suppose $F_1 = \{f | f : A \to$
B be function such that $f(1) = 1$ & $f(2) \neq 2$ } & $F_2 = \{f | f : A \to B$ be a one − one function such that $f(1) = 1$ & $f(2) \neq 2$ } then the cardinality of sets F_1 & F_2 respectively are ?

(1) 9, 9 (2) 20, 10
(3) 20, 9 (4) 18, 9

Solution:- Given $|A| = 3$ & $|B| = 5$
(i) Given $f(1) = 1$ & $f(2) \neq 2$
i.e. image of element 1 is fixed
$f(2)$ have 4 choices (i.e. 1, 3, 4, 5)
$f(3)$ have 5 choices (i.e. 1, 2, 3, 4, 5)
Hence, total number of functions in F_1 are $1 \times 4 \times 5 = 20$

(ii) Here $n = |B| = 5 > 3 = m = |A|$
$f(1) = 1$ & $f(2) \neq 2$
$\therefore$ $f(2)$ has three choices (i.e. 3,4,5)
& $f(3)$ has three choices of other then 1 and $f(2)$.
So,
Total number of functions in set F_2 is $1 \times 3 \times 3 = 9$
Hence, Option (3) is correct option

Q. The number of one-one function from set A to set B where $A = \{0, 1, 2, 3\}$ & $B = \{1, 2, 3, 4, 5\}$ are

(a) 60 (b) 40
(c) 100 (d) 120

Solution:- Here $|A| = m = 4$ and $|B| = n = 5$
$\Rightarrow m \leq n$
$\therefore$ Total number of one-one functions from A to B is ${}^n p_m$
$= {}^5 p_4 = \frac{5!}{(5-4)!} = \frac{5!}{1!} = 5! = 120$

1.4.2 Many-one function:

A function $f : A \to B$ is said to be many to one function, if there is at least two elements in the domain A has same image in B.

Note: A function that is not one-one is called to be many-one function.

Counting number of many to one function:
If $|A| = m$ & $|B| = n$ then
Total number of many-one function =
Total number of functions – Total number of one-one function
If $m > n$, then
Number of many to one function = $n^m - o = n^m$
If $m \leq n$, then
Number of many to one function = $n^m - {}^n P_m$

1.4.3 Onto (surjective) function:
A function $f: A \to B$ is said to be onto if each element of B has atleast one preimage in A.

Preimage: If f is function such that $y = f(x)$. Then we say
(i) y is image of x under f
(ii) x is preimage of y under f

Note: $f: A \to B$ is said to be onto if $\forall\, y \in B, \exists\, x \in A$ such that
$f(x) = y$

Counting number of onto functions form A to B:
1. If $|A| = m$ & $|B| = n$, then the number of onto functions from A to B is
$$= \begin{cases} n^m - \left[\sum_{r=1}^{n} (-1)^{r+1}\, n_{C_r} (n-r)^m\right], & m \geq n \\ 0 & m < n \end{cases}$$
Note:
(i) If range set is proper subset of B then f cannot be an onto function.
(ii) If $R(f) = B$, then f is onto

Examples:
The number of onto functions (surjective functions) from set $X = \{1, 2, 3, 4\}$ to set $Y = \{a, b, c\}$ is
(a) 36 (b) 64
(c) 81 (d) 72
Solution: $|X| = m = 4, |Y| = n = 3$
$\therefore$ total number of onto functions from X to Y
$= n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m - {}^nC_3(n-3)^m + \cdots$
$= 3^4 - {}^3C_1(3-1)^4 + {}^3C_2(3-2)^4 - {}^3C_3(3-3)^4 + \cdots$
$= 81 - 3(16) + 3(1)^4 = 81 - 48 + 3 = 36$
Total number of onto functions are 36
Hence, option (a) is correct.

1.4.4 Into function:
A function $f: A \to B$ is said to be into, if there exist an element in B which does not have preimage in A.
i.e. $f(A) \subsetneq B$

NOTE:
(i) A function which is not onto is an into function.
(ii) Number of into functions + number of onto functions = Total number of functions

Important Note:
(i) If $|A| = m$ & $|B| = n$, then total number of constant functions are n.
(ii) If $|A| = m$ & $|B| = 2$, then any function from A to B is into iff it is constant function.
$\Rightarrow$ Total number of into function from A to B is 2
Total number of functions from A to B is 2^m
$\therefore$ Total number of onto functions from A to B is $2^m - 2$.

1.4.5 One to one and onto OR Bijective OR invertible function:
A function $f: A \to B$ is bijective if it is one-one function and onto function.

Note:
(1) If $f: A \to B$ be a bijective function
$\Rightarrow$ (i) f is one-one $\Rightarrow |A| \leq |B|$
& (ii) f is onto $\Rightarrow |A| \geq |B|$
$\therefore$ If $f: A \to B$ be a bijective function then $|A| = |B|$

Converse need not be true.
i.e. if $|A| = |B| \not\Rightarrow f: A \to B$ is a bijective function

Example: If $f: [0, 1] \to [0, 1]$ such that $(x) = 1\ \forall x \in A$.
$\nexists$ any $x \in [0, 1]$ such that $f(x) = 0$
$\Rightarrow f$ is not onto function.
$\Rightarrow f$ is not a bijective function.

2. If $f: A \to B$ be a function where $|A| = |B| = n$ then f is one-one iff f is onto iff f is bijective
The above statement is not necessarily true when the cardinality of A & B are infinite.

Ex. $f: \mathbb{N} \to \mathbb{N}$ such that $f(n) = n + 1$
Here $|\mathbb{N}| = |\mathbb{N}|$ but f is not onto
$\because \nexists$ any $n \in \mathbb{N}$ such that $f(n) = 1$

Counting Number of bijective functions:
(i) If $|A| = |B| = n$, then number of bijective functions from A to B is equals to $n!$
(ii) If $|A| = |B| = n$, then number of functions from A to B which is neither one-one nor onto is equals to $n^n - n!$

Example:
If $A = \{1,2,3.4,5\}$ & $B = \{a, b, c, d, e\}$. Then
(a) The number of bijective functions
Answer: 5!
(b) The number of function which is one-on e but not onto
Answer: 0
(c) The number of function which is onto but not noe-one
Answer: 0
(d) The number of function which is neither one-one nor onto
$5^5 - 5!$

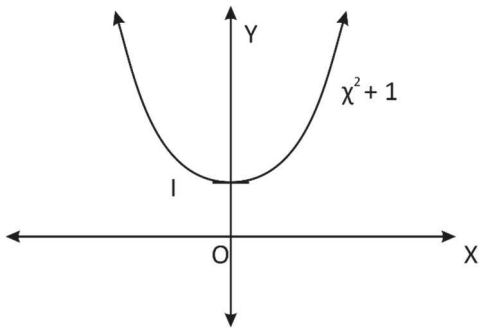
Note:
If $f: \mathbb{R} \to \mathbb{R}$ is a function then
(1) f is not one-one if graph of f cuts some line parallel to x –axis at least two points.
(2) f is one-one if graph of f cuts every lines parallel to x –axis at most one point.
(3) f is not onto if graph of f cuts some lines parallel to x –axis at no point.
(4) f is onto if graph of f cuts every lines parallel to x –axis at atleast one point.

Examples:
Q. If the functions $f, g: \mathbb{Z} \to \mathbb{Z}$ defined as $(n) = 3n + 2$ & $g(n) = n^2 - 5$, then
(1) Neither f nor g is one-one function
(2) The function f is one-one but not g

(3) The function g is one-one but not f
(4) Both f & g are one-one functions
Solution:-
(i) $f(n) = 3n + 2$
Let $n_1, n_2 \in \mathbb{Z}$ such that $f(n_1) = f(n_2)$
$\Rightarrow 3n_1 + 2 = 3n_2 + 2$
$\Rightarrow 3n_1 = 3n_2$
$\Rightarrow n_1 = n_2$
$\Rightarrow f$ is one-one function
(ii) $g(n) = n^2 - 5$
Here for any $n \in \mathbb{Z}$
$g(n) = n^2 - 5$ & $g(-n) = (-n)^2 - 5 = n^2 - 5$
But $n \neq -n$
$\Rightarrow g$ is not one-one function
Hence, option (2) is correct

Q. Let $f: X \to X$ such that $f(f(x)) = x$ for all $X \in X$. Then
(1) f is one-one and onto
(2) f is one-one but not onto
(3) f is onto but not one-one
(4) f need not be either one-one or onto
Solution:- for any $x \in X$, $f(f(x)) = x$
Let $x_1, x_2 \in X$ such that $f(x_1) = f(x_2)$
Apply f on above equation
$\Rightarrow f(f(x_1)) = f(f(x_2))$ $(\because f$ is a function$)$
$\Rightarrow x_1 = x_2$
$\Rightarrow f$ is one-one function
Onto :
(i) for any $x \in X$ (codomain), there is an element $f(x) \in X$ such that $f(f(x)) = x$
i.e. for any $i \in X$ (Codomain), $f(i)$ is a Preimage of i
Hence, f is onto function
OR,
If $f(g(x)) = x \quad \forall x \in X$ then f is onto & g is one-one function.
Consider $f(f(x)) = x$
$\Rightarrow f$ is onto & f is one-one function
Hence f is bijective map.
Thus, option (1) is correct.

Q. The function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = (x^2 + 1)^{2021}$ is
(1) Onto but not one-one
(2) One-one but not onto
(3) Both one-one & onto
(4) Neither one-one nor onto
Solution:- Here $f(x) = (x^2 + 1)^{2021}$
$f(-x) = ((-x)^2 + 1)^{2021}$ but $x \neq -x$
So, f is not one-one function
Consider $\forall x \in \mathbb{R}, x^2 + 1 \geq 1$
$\Rightarrow (x^2 + 1)^{2021} \geq (1)^{2021} = 1$
Hence



For any $y \in (-\infty, 1), \nexists$ any $x \in \mathbb{R}$ such that $f(x) = y$
Thus, f is not onto function.

Q. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{Z}_{\geq 0}$ denotes the set $\{0, 1, 2, 3, \ldots.\}$. Consider the map $f: \mathbb{Z}_{\geq 0} \times \mathbb{Z} \to \mathbb{Z}$ Given by $f(m, n) = 2^m.(2n + 1)$. Then the map f is
(a) Onto (surjective) but not one-one (injective)
(b) One-one (injective) but not onto (surjective)
(c) Both one- one and onto
(d) Neither one-one nor onto
Solution :
Given, $f: \mathbb{Z}_{\geq 0} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ defined by $f((m, n)) = 2^m(2n + 1)$
Let $(m_1, n_1), (m_2, n_2) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$
Then, $f((m_1, n_1)) = f((m_2, n_2))$
$\Rightarrow 2^{m_1}(2n_1 + 1) = 2^{m_2}(2n_2 + 1)$
$\Rightarrow$ If $m_1 = m_2 \Rightarrow 2n_1 + 1 = 2n_2 + 1 \Rightarrow n_1 = n_2$
If $m_1 \neq m_2$, Then, without loss of generality we can consider that $m_1 > m_2$
$\Rightarrow 2^{m_1 - m_2}(2n_1 + 1) = (2n_2 + 1)$,
Which is a contradiction because term on left hand side is a multiple of 2 i.e even number and term on right hand side is an odd number.
So, $m_1 = m_2$ and $n_1 = n_2$
$\therefore f$ is an one-one function.
As, $0 \in \mathbb{Z}$ and $f(m, n) = 2^m(2n + 1) \neq 0 \; \forall (m, n) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}$
$(\because 2^m(2n + 1) = 0 \Rightarrow n = -\frac{1}{2} \notin \mathbb{Z})$
$\Rightarrow f$ is not an onto function.
Hence option (b) is correct

Q. Let $A = \{x^2 : 0 < x < 1\}$ and $B = \{x^3 : 1 < x < 2\}$ Which of the following statements is true?
(a) There is a one to one, onto function from A to B
(b) There is no one to one, onto function from A to B taking rational to rational.
(c) there is no one to one function from A to B which is onto
(d) There is no onto function from A to B Which is one to one

Solution: $A = \{x^2 : 0 < x < 1\}$ and $B = \{x^3 : 1 < x < 2\}$
A and B are open intervals and in metric space (R, d) open intervals are homeomorphic
So, there exists a homeomorphism of A onto B.
∴ there is one-one and onto function from A to B.
∴ option (a) is correct.

Q. How many onto function from set A = { 1,2,3,4,5,6,7,8,9,10} to the set B = { a, b }
(a) 1024 (b) 1022
(c) 100 (d) 98
Answer : (2)

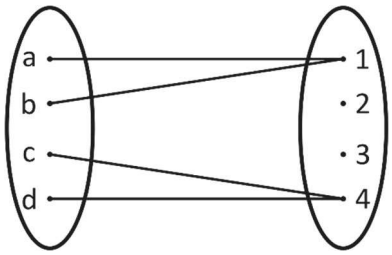
Q. Let $f(n) = n^3 - 3n$ defined on set of integers is injective function. [T/F]
Answer : False

Note:
(1) If $f: A \to B$ is an inverse function then $f^{-1}: B \to A$ is also invertible function where $f^{-1}(y) = x$, whenever $f(x) = y$
(2) Let $f: A \to B$ is bijective function then
(i) $f \circ f^{-1} = I$ where $f \circ f^{-1}: B \to B$
(ii) $f^{-1} \circ f = I$ where $f^{-1} \circ f: A \to A$

Result:
1. If $f: X \to Y$ is one-one function & $A, B \subseteq X$, then
(i) $f(A \cup B) = f(A) \cup f(B)$
(ii) $f(A \cap B) = f(A) \cap f(B)$

2. If $f: X \to Y$ be a function such that f^{-1} exist & $A, B \subseteq Y$ then
(i) $f^{-1}(A \cup B) = f^{-1}(A) \cup f^{-1}(B)$
(ii) $f^{-1}(A \cap B) = f^{-1}(A) \cap f^{-1}(B)$
(iii) $f^{-1}(A^c) = (f^{-1}(A))^c$

Example:-
1. If $f: X \to Y$ is a function & $A, B, C \subseteq X$. If $f(\phi) = \phi$ then which of the following is/are true?
(1) $f(A \cup B) = f(A) \cup f(B)$
(2) $f(A \cap B) = f(A) \cap f(B)$
(3) $f(A \cap B) \subseteq f(A) \cap f(B)$
(4) $f(A \backslash B) = f(A) \backslash (B)$
Solution:
(a) $A, B \subseteq A \cup B \Rightarrow f(A), f(B) \subseteq f(A \cup B) \Rightarrow f(A) \cup f(B) \subseteq f(A \cup B)$
$x \in A \cup B \Rightarrow f(x) \in f(A \cup B)$
Hence $f(A \cup B) \subseteq f(A) \cup f(B)$
$\Rightarrow f(A \cup B) = f(A) \cup f(B)$



(b)
Let $X = \{a, b, c, d\}$, $Y = \{1,4\}$ and let $f: X \to Y$ such that $f(a) = 1, f(b) = 1, f(c) = 4, f(d) = 4$
$A = \{a, c\} \Rightarrow f(A) = \{1,4\}$
$B = \{b, d\} \Rightarrow f(B) = \{1,4\}$
$\Rightarrow f(A) \cap f(B) = \{1,4\}$
But $A \cap B = \phi$
$\Rightarrow f(A \cap B) = \phi$
$\Rightarrow f(A \cap B) \neq f(A) \cap f(B)$
So, Option (b) is incorrect.
If $x \in A \cap B \Rightarrow x \in A \,\&\, x \in B \Rightarrow f(x) \in f(A) \,\&\, f(x) \in f(B)$
$f(x) \in f(A \cap B)$
$\Rightarrow f(A \cap B) \subseteq f(A) \cap f(B)$
Hence, option (c) is correct

Let $X = \{a, b, c, d\}$, $Y = \{1,4\}$
$A = \{a, b\}$ and $B = \{c, d\}$ and let $f: X \to Y$ such that $f(x) = 1 \; \forall x \in \{a, b, c, d\}$
Then
$f(A) = \{1\}$ and $f(B) = \{1\}$
Therefore, $f(A \backslash B) = f(\{a, b\}\{c, d\}) = f(\{a, b\}) = \{1\}$
And $f(A) \backslash (B) = \{1\} \backslash \{1\} = \phi$
Hence, option (d) is incorrect

1.4.6 Cantor's Theorem:
There is no onto map from any non-empty set to its power set.
Note:- There exist a one-one map from set A to $P(A)$
Since $f: A \to P(A)$ is not an onto map.
i.e. potential of power set of any set is more than set itself.
i.e. (potential of $P(\mathbb{N})$ is more than $\mathbb{N}$)
Remark:-
For any infinite set A, $|A| \ll |P(A)| \quad (i.e\ |\mathbb{Z}| < |P(\mathbb{Z})|, |\mathbb{Q}| < |P(\mathbb{Q})|, etc)$
∴ We need to define symbols to denote cardinality of infinite sets of different potential.
These symbols are called transfinite cardinals.
$|\mathbb{N}| = \aleph_0$ (aleph null, aleph nought)
$|P(\mathbb{N})| = c$ (continum)

1.5 Similar /Equivalent Sets:

1.5.1 Definition:
Two sets A & B are said to be equivalent or similar if there is one to one correspondence between their elements.
i.e. there exist a bijective map from A to B and it is denoted by $A \sim B$.

When two infinite sets are similar, the same transfinite cardinals are used to denote their cardinality & conversely.

The Initial segment:

Let $n \in \mathbb{N}$

The initial segment of the natural numbers determined by n $\{1, 2, 3, \dots, n\}$ & is denoted by $\mathbb{N}_{\leq n}$ or S_n

(We will use S_n)

Two finite sets A & B are similar iff $|A| = |B|$

Two finite set A & B are similar iff both A & B are similar to S_n for some $n \in \mathbb{N}$

Example:

Which of the following sets equivalent to $S_4 = \{1, 2, 3, 4\}$

(a) $A_1 = \{a, b, c, d\}$

(b) $A_2 = \{2,3,4,1\}$

(c) $A_3 = \{\{1\}, \{2\}, \{3\}, \{4\}\}$

(d) $A_4 = \{\{1,2\}, \phi, \{2,3\}, S_4\}$

(e) $A_5 = \{\mathbb{N},\ P(\mathbb{N}),\ \mathbb{R},\ \mathbb{R} \times \mathbb{R}\}$

Solution : All sets are similar to S_4

$\because |A_1| = |A_2| = |A_3| = |A_4| = |A_5| = 4$

Note :

i) There does not exist any transfinite Cardinal number which is strictly less than $\aleph_0$

ii) There does not exist any transfinite Cardinal number between $\aleph_0$ and c.

iii) There is infinitely many transfinite Cardinal number which is strictly greater than c.

Examples :-

Which of the following is/are correct ?

(1) $\mathbb{N} \sim k\mathbb{N}$, whose $k \in \mathbb{R} - \{0\}$

(2) $\mathbb{N} \sim \mathbb{Z}$

(3) $\mathbb{N} \sim \mathbb{N} \times \mathbb{N}$

(4) None of the above

Solution:-

(1) Define $f: \mathbb{N} \to k.\mathbb{N}$ such that

$$f(n) = kn$$

Clearly, f is well defined

i) Let $f(n_1) = f(n_2)$
$\Rightarrow kn_1 = kn_2$
$\Rightarrow n_1 = n_2$
$\Rightarrow f$ is one one map

ii) for any $kn \in k\mathbb{N}$, take $n \in \mathbb{N}$ such that $f(n) = kn$
$\Rightarrow f$ is onto map
$\Rightarrow f$ is bijective map

Hence, $\mathbb{N} \sim k.\mathbb{N}\ \ \forall\ k \in \mathbb{R} - \{0\}$

(2) Define $f: \mathbb{N} \to \mathbb{Z}$ such that

$f(n) = (-1)^n \left[\frac{n}{2}\right]$

Or, $f(n) = \begin{cases} -\left(\frac{n-1}{2}\right) & \text{if } n \text{ is odd} \\ \frac{n}{2} & \text{if } n \text{ is odd} \end{cases}$

$f(1) = 0, f(2) = 1, f(3) = -1, f(4) = 2, f(5) = -2$

$\therefore f$ is a bijective map

Hence, $\mathbb{N} \sim \mathbb{Z}$

Result :- $'\ \sim\ '$ is an equivalence relation

Proof:

i) Let A be any set
Define $f: A \to A$ such that $f(x) = x$
$\Rightarrow f$ is bijective
$\Rightarrow A \sim A\ \Rightarrow\ \sim$ is reflexive

ii) Let $A \sim B\ \Rightarrow \exists$ a bijective map $f: A \to B$
Take, $f^{-1}: B \to A$
As f is bijective map $\Rightarrow f^{-1}$ is also bijective map
$\Rightarrow B \sim A$
$\Rightarrow\ '\sim'$ is a symmetric.

iii) Let $A \sim B$ & $B \sim C$
then $\exists$ two bijective maps $f: A \to B$ & $g: B \to C$
As f & g are bijective maps
$\Rightarrow gof$ is also bijective map, where $gof: A \to C$
$\Rightarrow A \sim C$
$\Rightarrow\ '\sim\ '$ is transitive
$\Rightarrow'\ \sim\ '$ is an equivalence relation.

1. Q. Is $\mathbb{N} \sim$ [0,1] ?

Solution:-

Suppose $\mathbb{N} \sim [0,1] \Rightarrow \exists$ one-one onto function $'f'$ from $\mathbb{N}$ to $[0,1]$

Let the mapping is as follows

$$\begin{cases} f(1) = 0.a_{11}a_{12}a_{13} \dots\dots\dots a_{1n} \dots\dots. \\ f(2) = 0.a_{21}a_{22}a_{23} \dots\dots\dots a_{2n} \dots\dots. \\ f(3) = 0.a_{31}a_{32}a_{33} \dots\dots\dots a_{3n} \dots\dots. \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \\ f(n) = 0.a_{n1}a_{n2}a_{n3} \dots\dots\dots a_{nn} \dots\dots\dots \end{cases}$$

Take a number x in $[0,1]$ such that

$x = 0.b_1b_2b_3 \dots\dots b_n \dots\dots$such that $b_i \neq a_{ii}$

$\Rightarrow x$ is different from $f(i)\ \ \forall\ i \in \mathbb{N}$

$\Rightarrow \nexists$ any $i \in \mathbb{N}$ such that $f(i) = x$

$\Rightarrow f$ is not an onto map.

Hence, our assumption is wrong

$\Rightarrow \mathbb{N} \nsim [0,1]$.

1.6 Countable & Uncountable Sets

1.6.1 Countably Infinite set:

A set S is said to be countably infinite set or denumerable set or enumerable set if it is similar to $\mathbb{N}$.

i.e. S is countably infinite iff $S \sim \mathbb{N}$.

Examples:

i) $\mathbb{N}, \mathbb{Z}, k.\mathbb{N}, k.\mathbb{Z}$ (where $k \in \mathbb{R} - \{0\}$)

ii) $\mathbb{R}$ is not countable infinite set.

Reason: [0,1] $\subseteq \mathbb{R} \Rightarrow |[0,1]| \leq |\mathbb{R}|$

But , we know that $\mathbb{N} \nsim [0,1]$ implies $|[0,1]| > \aleph_0$

$\Rightarrow |\mathbb{R}| > \aleph_0$

$\Rightarrow \mathbb{R}$ is not countably infinite set.

Observation :- A set S is countably infinite $\Leftrightarrow |S| = \aleph_0$

1.6.2 Countable set:

A set S is said to be countable set if it is either finite or countably infinite.

Ex:

i) Any finite set is countable set.

ii) Any set similar to set of natural number is countable set.

iii) Empty set is countable set ($|\emptyset| = 0 \Longrightarrow \emptyset$ is finite set.)

1.6.3 Uncountable set:

A set S is said to be uncountable set if it is not countable.

Observation:

If the set 'S' has cardinality C (continum) then S is uncountable. Converse need not be true.

Example: $P(P(P(\mathbb{N})))$ is an uncountable set, but | P(P(P(ℕ)))| $> 2^c > c$

Ex.

1. $[a,b]$, $[a,b)$, $(a,b]$, (a,b) are uncountable sets (a$< b$)
2. Set of real number is uncountable set.
3. $P(\mathbb{N})$ is an uncountable set

Algebra of Transfinite cardinal numbers:

Let $\aleph_0$ denotes the cardinality of $\mathbb{N}$ & C denotes the cordinality of $P(\mathbb{N})$, then

1) $\aleph_0 + \aleph_0 + \aleph_0 + \cdots + \aleph_0 = \aleph_0$
2) $C + C + \cdots + C = C$
3) $n + \aleph_0 = \aleph_0 \ \forall\, n \in \mathbb{N}$
4) $\aleph_0 + C = C$, $\aleph_0 \cdot C = C$
5) $\aleph_0 \cdot \aleph_0 \cdots \aleph_0 = \aleph_0^n = \aleph_0 \ \forall\, n \in \mathbb{N}$
6) $C \cdot C \cdots C = C^n = C \ \forall\, n \in \mathbb{N}$
7) $2^{\aleph_0} = C > \aleph_0$
8) $\aleph_0^{\aleph_0} = 2^{\aleph_0} = C$
9) $\aleph_0^C = 2^C$
10) $C^{\aleph_0} = C$

viii) $n < \aleph_0 < C \ \forall\, n \in \mathbb{N}$

Q: Let A be any subset of real number. Then which of the following is possible?

a) $P(A)$ is an infinite set

b) $P(A)$ is empty set

c) $P(A)$ is countably infinite set

d) $P(A)$ is countable set

e) $P(A)$ is uncountable set

Solution:

a) If $A = \mathbb{N} \Rightarrow |P(A)| = 2^{\aleph_0} = C$
$P(A)$ is an infinite set.

b) for any non-empty set A such that $|A| \geq 1$
$\Rightarrow |P(A)| \geq 2^1 = 2$
If $A = \emptyset \Rightarrow P(A) = \{\emptyset\} \Rightarrow |P(A)| = 1$
$\therefore$ for any set A, $|P(A)| \geq 1$

c)

i) If A is finite set $\Longrightarrow |A| = n$ for some $n \in \mathbb{N}$
$\Rightarrow |P(A)| = 2^n$ (finite)
i.e If set 'A' is finite then $P(A)$ is also finite set.

ii) If A is infinite set $\Rightarrow |A| \geq \aleph_0$
$\Longrightarrow |P(A)| \geq 2^{\aleph_0} = C$
$\Rightarrow$ If A is an infinite set, then cardinality of power set of A is at least continum.
Thus, For any set A, $|P(A)| \neq \aleph_0$
i.e for any set A, $P(A)$ cannot be countably infinite.

d) As $P(A)$ is finite for any finite set A
$\Rightarrow P$ is countable set.

e) $A = \mathbb{N} \Rightarrow |P(A)| = 2^{\aleph_0} = C$
$\Rightarrow P(A)$ is uncountable

1.6.4 Result:

1) Every infinite set has a countably infinite Subset.
2) Every infinite set is similar to at least one of it's proper subset.
3) If S is infinite subset of $\mathbb{R}$ & X is any finite subset of S, then $|S| = |S - X|$

Example : Show that $[a,b] \sim [c,d]$, $(a,b) \sim (c,d)$ & $[a,b] \sim (a,b)$

Solution: Consider, two sets $[a,b]$ & $[c,d]$, $a < b$, $c < d$

There is a line which map $[a,b]$ to $[c,d]$

$$y = \left(\frac{d-c}{b-a}\right)(x-a) + c$$

Which is one-one & onto map $\Rightarrow [a,b] \sim [c,d]$

By using the same map, we get $(a,b) \sim (c,d)$

We know that If S is infinite subset of $\mathbb{R}$ & X is any finite subset of S, then $|S| = |S - X|$

If $S = [a,b]$ and $X = \{a,b\}$ then $|S| = |S - X|$

i.e. $|[a,b]| = |(a,b)|$

Example: Show that $A = \left(\frac{-\pi}{2}, \frac{\pi}{2}\right)$ & $B = \mathbb{R}$ are similar

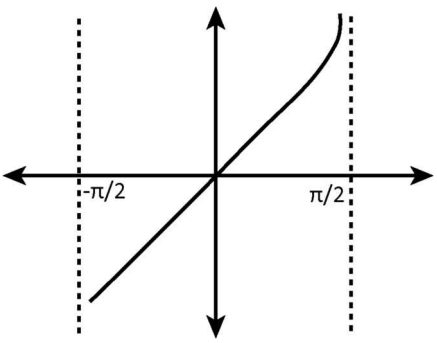
Define $f: A \to B$

Such that $f(x) = tan$

Clearly, f is bijective map.

$\Rightarrow$ A ~ B

$\Rightarrow \mathbb{R} \sim \left(\frac{-\pi}{2}, \frac{\pi}{2}\right) \sim \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$



Some important results on Countability :

1. Countable union of countable set is countable.
2. Family of all finite subsets of countable set is countable.
3. Family of all infinite subset of countably infinite set is uncountable.
4. Finite Cartesian product of countable set is countable.
5. Infinite Cartesian product of countable set $A_1, A_2, \ldots$ (where $|A_i| \geq 2$ for infinitely many i) is an uncountable set

Example: Show that Set of rational number is countable set.

Solution: The given set $\mathbb{Q}$ can be written as

$\mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$

Consider the following set

$E_1 = \left\{\frac{1}{n} | \, n \in \mathbb{N}\right\}$

$E_2 = \left\{\frac{2}{n} | \, n \in \mathbb{N}\right\}$

$E_3 = \left\{\frac{3}{n} | n \in \mathbb{N}\right\} \ldots$

$E_k = \left\{\frac{k}{n} | \, n \in \mathbb{N}\right\}$

Here each E_k is countable set $\forall \, k \in \mathbb{N}$

(Define $f: \mathbb{N} \to E_2$ such that $f(n) = \frac{k}{n}$)

We know that, countable union of countable set is countable.

$\Rightarrow \bigcup_{k=1}^{\infty} E_k$ is countable where $\bigcup_{n=1}^{\infty} E_k = \mathbb{Q}^+$

$\Rightarrow \mathbb{Q}^+$ is countable set

Define $f: \mathbb{Q}^+ \to \mathbb{Q}^-$ such that $f(x) = -x$

Clearly, f is bijective function.

$\Rightarrow \mathbb{Q}^-$ is also countable set

Hence as $\mathbb{Q}^+, \mathbb{Q}^- \{0\}$ all are countable set.

$\Rightarrow \mathbb{Q} = \mathbb{Q}^+ \cup \mathbb{Q}^- \cup \{0\}$ is countable set

Now, $\mathbb{R} = \mathbb{Q} \cup \mathbb{Q}^c$

If $\mathbb{Q}^c$ is countable set

$\Rightarrow \mathbb{Q} \cup \mathbb{Q}^c$ is also Countable ($\because \mathbb{Q}$ is Countable)

$\Rightarrow \mathbb{R}$ is Countable set

Which is a contradiction.

Hence, $\mathbb{Q}^c$ is an Uncountable Set.

Q. Which of the following set is countable?

1] $S_1 = \{f | f: \{1,2\} \to \mathbb{N}$ **is a function}**

Solution: Set S_1 contains all functions f from $\{1,2\}$ to $\mathbb{N}$

Thus, $|S_1| = |\mathbb{N}|^{|\{1,2\}|} = \aleph_0^2 = \aleph_\circ \times \aleph_\circ = \aleph_\circ$

$\Rightarrow S_1$ is countable set.

2) $S_2 = \{f \, | f: \mathbb{N} \to \{1,2\}$ **is a function}**

Solution: Set S_2 contains all functions f from $\mathbb{N}$ to $\{1,2\}$

Thus, $|S_2| = |\{1,2\}|^{|\mathbb{N}|} = 2^{\aleph_\circ} = C$

$\Rightarrow S_2$ is an uncountable set.

3] $S_3 = \{f | f: \{1,2\} \to \mathbb{N}$ **is a function** $\& \, f(1) \leq f(2)\}$

Solution: We know that S_1 is countable set.

Here, $S_3 \subseteq S_1$

A subset of a countable set is countable set.

$\Rightarrow S_3$ is countable set.

4] $S_4 = \{f | f: \mathbb{N} \to \{0,1\}$ **is a function &** $f(1) \leq f(2)\}$

Solution: Here the possible choices for $n = 1$ & $n = 2$ are

i) $f(1) = 1, f(2) = 1$

ii) $f(1) = 0, f(2) = 1$

iii) $f(1) = 0, f(2) = 0$

The choices for all other elements 3,4,5,6,…… are {0,1}

Total number of functions $f: \mathbb{N} \to \{0,1\}$ such that $f(1) \leq f(2)$ are

$3 \times 2 \times 2 \times 2 \times 2 \ldots\ldots\ldots\ldots\ldots = 3 \times 2^{\aleph_\circ} = C$

$\Rightarrow S_4$ is an uncountable set.

5] $S_5 = \{f | f: \mathbb{N} \to \{1,2,3 \ldots . . n\}$ **be a function}**

Solution: Set S_5 contains all functions f from $\mathbb{N}$ to $\{1,2,3 \ldots . . n\}$

Thus, $|S_5| = |\{1,2,3 \ldots . . n\}|^{|\mathbb{N}|} = n^{\aleph_\circ} = C$

$\Rightarrow S_5$ is an uncountable set.

6] $S_6 = \{f | f: \{1,2,3, \ldots . n\} \to \mathbb{N}$ **be a function}**

Solution: Set S_6 contains all functions f from $\{1,2,3, \ldots . n\}$ to $\mathbb{N}$

Thus, $|S_6| = |\mathbb{N}|^{|\{1,2,3,\ldots,n\}|} = \aleph_0^n = \aleph_\circ$

$\Rightarrow S_6$ is countable set.

7] $S_7 = \{a_n | a_n$ **is a sequence** $\& \, a_n \in \{0,1\} \, \forall n \in \mathbb{N}\}$

Solution: We know that,

Sequence is a special type of function having domain set of natural numbers.

Set S_7 contains all functions f from $\mathbb{N}$ to $\{0,1\}$

$|S_7| = |\{0,1\}|^{|\mathbb{N}|} = 2^{\aleph_\circ} = C$

Solved Examples :

1. Which of the following sets is uncountable?

(a) $\{x \in \mathbb{R} | \log(x) = \frac{p}{q}$ for some $p, q \in \mathbb{N}\}$

(b) $\{x \in \mathbb{R} | (\cos(x))^n + (\sin(x))^n = 1 \; for \; some \; n \in \mathbb{N}\}$

(c) $\left\{x \in \mathbb{R} \middle| x = \log\left(\frac{p}{q}\right) \; for \; some = p, q \in \mathbb{N}\right\}$

(d) $\{x \in \mathbb{R} | \cos(x) = \frac{p}{q} \; for \; some \; p, q \in \mathbb{N}\}$

Solution:

(a) we have log $x = p/q$

$\Rightarrow x = e^{p/q}$

$= e^r$

$p, q \in \mathbb{N}$

$r \in \mathbb{Q}$

$\because |\mathbb{Q}| = \aleph_0$ i.e. $\mathbb{Q}$ is countable set.

$\therefore$ choices of e^r is also countable.

Therefore (a) is incorrect

(b) $(\cos x)^n + (\sin x)^n = 1; n \in \mathbb{N}$

Take $n = 2, cos^2x + sin^2x = 1 \quad \forall x \in \mathbb{R}$

$\Rightarrow S = \{x \in \mathbb{R} | cos^2x + sin^2x = 1\} = \mathbb{R}$

$\therefore |S| = |\mathbb{R}| = C$

$\Rightarrow$ option (b) is correct

(c) $x = \log\left(\frac{p}{q}\right)$

$\Rightarrow e^x = p/q$, for some $p, q \in \mathbb{N}$

$= p/q \in \mathbb{Q}$

$\because |\mathbb{Q}| = \aleph_0$

Option (c) is incorrect.

(d) $\cos x = \frac{P}{q} = r \in \mathbb{Q}$, for some $p, q \in \mathbb{N}$

$|\mathbb{Q}| = \aleph_0$

Choices for x such that $\cos x = r$ also countable

Option (d) is incorrect.

2. Let $S = \left\{(x, y) | \, x^2 + y^2 = \frac{1}{n^2}\right.$, where $n \in \mathbb{N}$ and either $x \in \mathbb{Q}$ or $y \in \mathbb{Q}\}$. Here $\mathbb{Q}$ is the set of rational numbers and $\mathbb{N}$ is the set of positive integers. Which of the following is true?

(a) S is a finite non-empty set

(b) S is countable

(c) S is uncountable

(d) S is empty

Solution:

$S = \{(x, y) | x^2 + y^2 = \frac{1}{n^2}$ where $n \in \mathbb{N}$ and either $x \in \mathbb{Q}$ or $y \in \mathbb{Q}\}$.

Solving equation $x^2 + y^2 = \frac{1}{n^2}$ for y with $n = 1$, we get $y = \pm\sqrt{1 - x^2}$

For any $x \in \mathbb{Q}$, we have at most two solution for y

In general, for $n = k$

$S_k = \left\{(x, y): x^2 + y^2 = \frac{1}{k^2}, x \in \mathbb{Q}\right\}$ is countable

$S' = \bigcup_{n \in \mathbb{N}} S_n$ is countable

Now for any $y \in \mathbb{Q}$,

We have at most two solutions for x, So for $n = k$ then the set

$S_k = \left\{(x, y): x^2 + y^2 = \frac{1}{k^2}, y \in \mathbb{Q}\right\}$ is countable

So, $S'' = \bigcup_{n \in \mathbb{N}} S_n$ is countable

$S = S' \cup S''$ is countable.

3. Which of the following is necessarily true for a function $f: X \to Y$?

(a) If f is injective, then there exists $g: Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$

(b) If f is surjective, then there exists $g: Y \to X$ such that $f(g(y)) = y$ for all $y \in Y$

(c) If f is injective and Y is countable then X is finite

(d) If f is surjective and X is uncountable then Y is countably infinite

Solution:

For option (a),

Let $X = \{0,1\}, Y = \{0,1,2\}$

Take, $f: X \longrightarrow Y$ be defined as $f(x) = x$

Clearly, f is injective but for function $g: Y \longrightarrow X$ such that $f(g(y)) = y$ for all $y \in Y$

There does not exist pre-image of 2 under $f \circ g$.

$\therefore$ Option (a) is incorrect.

For option (b),

As f is surjective

$\therefore f: X \longrightarrow Y$ for any $y \in Y$ there exists at least one $x \in X$ for which $f(x) = y$

Define $g: Y \longrightarrow X$ in such a way that $g(y) = x$

$f \circ g(y) = f(x) = y$

So, option (b) is correct.

For option (c),

Take, $X = \mathbb{N}, Y = \mathbb{N}$

Let $f: \mathbb{N} \longrightarrow \mathbb{N}$ defined by $f(x) = x$, clearly f is injective and Y is countable but X is also countable

Thus, option (c) is incorrect.

For option (d), Take, $X = \mathbb{R}, Y = \mathbb{R}$

Let $f: X \longrightarrow Y$ defined by $f(x) = x$

Clearly f is surjective and X is uncountable but Y is also uncountable

$\therefore$ option (d) is incorrect.

1.6.5 Algebraic & Transcendental Number:

If $P = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n / a_i \in \mathbb{Q}, n \in \mathbb{N} \cup \{0\}\} =$ the set of all polynomials with rational coefficients

Define $f: P \to E$ such that

Where $f: (a_0 + a_1x + \cdots + a_nx^n) = (a_0, a_1, \ldots, a_n)$

Clearly, f is one-one & onto

$\Rightarrow f$ is bijective

$\Rightarrow P \sim E$

Where $E = \underbrace{\mathbb{Q} \times \mathbb{Q} \times \mathbb{Q} \times \ldots \times \mathbb{Q}}_{n+1 \text{ times}}$

As $\mathbb{Q}$ is Countable and finite Cartesian product of countable set is countable.

$\Rightarrow E$ is Countable

$\Rightarrow P$ is Countable

The set P can be written as

$P = \{\alpha_1(x), \alpha_2(x), \alpha_3(x), \ldots, \alpha_n(x), \ldots\} \cup \mathbb{Q}$.

Where each $\alpha_i(x)$ is a polynomial over $\mathbb{Q}$.

$S_1 = \{a \in \mathbb{C} | \ \alpha(a) = 0\}$

S_1 contains all roots of one degree polynomial in P

By fundamental theorem of algebra, every nth degree polynomial has exactly n roots counting with multiplicity.

Thus, every one degree polynomial has exactly one root in $\mathbb{C}$.

Since, all one degree polynomials over $\mathbb{Q}$ is countable set.

Hence, S_1 is countable set.

$S_2 = \{a \in \mathbb{C} | \ \alpha(a) = 0\}$

S_2 contains all roots of two degree polynomial in P

Clearly, S_2 is countable set.

In general,

$S_n = \{\alpha \in \mathbb{C} | \ \alpha_n(x) = 0\}$

S_n contains all roots of nth degree polynomial in P

As finite Cartesian product of countable set is countable

Thus, S_n is countable.

We know that, countable union of countable set is countable.

$\Rightarrow S = \bigcup_{n=1}^{\infty} S_n$ is countable set.

Note: If $P = \{a_0 + a_1x + a_2x^2 + \cdots a_nx^n / a_i \in X, n \in \mathbb{N} \cup \{0\}\}$, then P is countable iff X is countable.

1.6.6 Definition:

1. A complex number is said to be algebraic over $\mathbb{Q}$ if it is zero of non-zero polynomial with rational coefficients.
   ($\because$ The set of all algebraic numbers in $\mathbb{C}$ is countable set) the set of all algebraic number in $\mathbb{R}$ is countable set)
2. A real number / Complex number is said to transcendental if it not algebraic.
   Clearly, the set of all algebraic number is countable.
   ($:\mathbb{R}$ =set of Algebraic numbers $\cup$ set of transcendental numbers)
   If set of transcendental number is countable then $\mathbb{R}$ is countable which a contradiction.
3. Every rational number is an algebraic number
4. Every irrational number need not be a transcendental number.
5. All finite subset of set of natural number is countable.
   Reason:
   Define
   $A_i = \{X \subseteq \mathbb{N} | \ |X| = i\}$
   $A_1 = \{X \subseteq \mathbb{N} | \ |X| = 1\}$
   $A_1 = \{\{1\}, \{2\}, \{3\}, \ldots\ldots, \{n\}, \ldots\ldots\}$
   $\Rightarrow A_1 \sim \mathbb{N} \Rightarrow A_1$ is countable
   $A_2 = \{X \subseteq \mathbb{N} | \ |X| = 2\}$
   $A_2 = \{\{a, b\} | \ a, b \in \mathbb{N}\}$
   Define,
   $f_2 : A_2 \to \mathbb{N} \times \mathbb{N}$ such that $f_2(x) = \begin{cases} (a, b), & a < b \\ (b, a), & b < a \end{cases}$ *where* $X = \{a, b\}$
   $\Rightarrow f$ is one-one & onto
   $\Rightarrow A_2 \sim \mathbb{N} \times \mathbb{N} \sim \mathbb{N} \Rightarrow A_2$ is Countable
   $A_n = \{X \subseteq \mathbb{N} | |X| = n\}$
   Clearly $A_n \sim \underbrace{\mathbb{N} \times \mathbb{N} \times -- \times \mathbb{N} \sim \mathbb{N}}_{n \text{ times}} \Rightarrow A_n$ is Countable
   $\therefore A = \bigcup_{i=1}^{\infty} A_i$ is a countable set
   Hence, set of all finite subset of natural number is countable.
   In general,
   Set of all finite subset of countable set is countable.
6. $P(\mathbb{N}) = \{X \subseteq \mathbb{N} | \ |X| \ is \ finite\} \cup \{X \subseteq \mathbb{N} | \ |X| \text{ is infinite}\}$
   $= S_1 \cup S_2$
   Where $P(\mathbb{N})$ is uncountable set & S_1 is Countable set
   $\Rightarrow S_2$ is uncountable set.
   i.e. The set of all infinite subset of Natural numbers is an uncountable set.
7. There is no Strictly decreasing function $f : \mathbb{N} \to \mathbb{N}$
8. The set of all decreasing function from $f : \mathbb{N} \to \mathbb{N}$ is countable set.
   Since, Every decreasing function $f : \mathbb{N} \to \mathbb{N}$ is an eventually constant function
9. The set of all monotonic function from $\mathbb{N}$ *to* $\mathbb{N}$ is uncountable $i.e. = \{f : \mathbb{N} \to \mathbb{N} | f \ is \ monotonic\}$
10. If $A = \{\langle a_n \rangle | \ a_n \text{is eventually constant in } X\}$, then A is countable iff X is countable.

1.7 Inequality:

1. For any real number a, b

1. $|a + b| \leq |a| + |b|$
2. $|a - b| \geq ||a| - |b||$
3. $\sqrt{a^2 + b^2} \leq |a| + |b|$
4. $\sqrt{|a| + |b|} \leq \sqrt{|a|} + \sqrt{|b|}$
5. If $a_1, a_2, \ldots\ldots, a_n$ be any real numbers, then $|a_1 + a_2 + \cdots + a_n| \leq +|a_2| + \cdots + |a_n|$
6. If $a_1, a_2, \ldots, a_n$ be any real numbers, then $|a_1.a_2, \ldots, a_n| = |a_1|.|a_2| \ldots\ldots |a_n|$
7. $|x + y| = |x| + |y| \quad , if \ xy \geq 0$
8. $|x + y| < |x| + |y| \quad , if \ xy < 0$
9. $\frac{|a+b|}{1+|a+b|} \leq \frac{|a|}{1+|a|} + \frac{|b|}{1+|b|}$

2. Cauchy – Schwarz Inequality:

1. If $a_1, a_2, \ldots . a_n\ , b_1, b_2, \ldots . b_n > 0$ &
$A = \sum_{i=1}^{n} a_i^2,\ B = \sum_{i=1}^{n} b_i^2$ & $C = \sum_{i=1}^{n} a_i b_i = a_1 b_1 + a_2 b_2 + \cdots . + a_n b_n$
Then $C^2 \leq A.B$
i.e $(\sum_{i=1}^{n} a_i b_i)^2 \leq (\sum_{i=1}^{n} {a_i}^2)\ (\sum_{i=1}^{n} {b_i}^2)$
Let $p > 1, \frac{1}{P} + \frac{1}{q} = 1\ , a \geq 0\ , b \geq 0$ then
$ab \leq \frac{a^p}{p} + \frac{b^q}{q}$ equality hold iff $a^p = b^q$

3. Holder's Inequality:

If $a_i b_i \geq 0, \forall\ i\ \in S_n = \{1, 2, \ldots . , n\}$ then
$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} {a_i}^p\right)^{1/P} \left(\sum_{i=1}^{n} {b_i}^q\right)^{1/q}\ , P > 1\ \& \frac{1}{P} + \frac{1}{q} = 1$

4. Minkowskis Inequality:

If $P \geq 1\ \&\ a_i, b_i \geq 0, \forall\ i\ \in S_n$ then

$$\left(\sum_{i=1}^{n} (a_i + b_i)^p\right)^{1/P} \leq \left(\sum_{i=1}^{n} {a_i}^p\right)^{1/P} + \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}$$

ASSIGNMENT

1. Let $f : \mathbb{R}\backslash\{0\} \to \mathbb{R}$ be defined by $f(x) = x + \frac{1}{x^3}$. On which of the following interval(s) is f one-one?
(A) $(-\infty, -1)$ (B) $(0, 1)$
(C) $(0, 2)$ (D) $(0, \infty)$

2. Let $f(x) = 2x^3 - 9x^2 + 7$. Which of the following is true?
(A) f is one-one in the interval [- 1, 1]
(B) f is one-one in the interval [2, 4]
(C) f is NOT one-one in the interval [- 4, 0]
(D) f is NOT one-one in the interval [0, 4]

3. Let X be a countably infinite subset of $\mathbb{R}$ and A be a countably infinite subset of X. Then the set $X\backslash A = \{x \in X\ |x \notin A\}$
(A) is empty
(B) is a finite set
(C) can be a countably infinite set
(D) can be an uncountable set

4. Let A and B be two sets having m and n many elements respectively, where $m < n; m, n \in \mathbb{N}$. The number of injective functions possible from A to B is:
(A) m^n (B) ${}^nP_m$
(C) ${}^nC_m$ (D) n^m

5. Let S be a set with n elements. Then the number of distinct objective mappings from $S\ to\ S$ is:
(A) n (B) n^2
(C) $n!$ (D) n^n

6. Which set is uncountable?
(A) The set of positive primes
(B) The set of integers
(C) The set of rational numbers
(D) The set of irrational numbers in [0, 1]

7. Which one of the following is necessarily true?
(A) If $f: \mathbb{Q} \to \mathbb{R}$ is an injective map, then $\overline{f(\mathbb{Q})}$ is uncountable
(B) There exists no surjective map from $\mathbb{N}$ onto $\mathbb{Q}$
(C) There exists a surjective map from $\mathbb{Q} \times \mathbb{R}$ onto $\mathbb{C}$
(D) None of the above

8. Consider the statements:
$[S_1]\ \overline{\left\{sin\left(\frac{1}{n}\right)\ |\ n \geq 1\right\}}$ is uncountable.
$[S_2]\ \overline{\left\{e^{\frac{1}{n}}|\ n \geq 1\right\}}$ is uncountable.
$[S_3]\ \overline{\{\pi^n|\ n \geq 1\}}$ is uncountable.
Which one of the following is the correct option?
(A) None of $[S_1]$, $[S_2]$ and $[S_3]$ is true
(B) All of $S_1]$, $[S_2]$ and$S_3]]$ are true
(C) Only $[S_1]$ is true
(D) Only $S_1$and $[S_2]$ are true

9. Let $S = \{a + b\sqrt{2} : a, b\ \in\ \mathbb{Z}\}\ \subset\ \mathbb{R}$.
Which of the following assertions about S is correct?
(A) For any maps $f: \mathbb{Q} \times \mathbb{Q} \to \mathbb{R}\ and\ g: \mathbb{R} \to S$, their composition $g\ 0\ f$ never surjective.
(B) There exists a surjective map from S onto R^2/z^2,
(C) There exists an injective map from R/Q into S.,
(D) For any two maps $f: S \to \mathbb{R}\ and\ g: \mathbb{R} \to \mathbb{R}^3$, their composition of gof has a countable range.

10. For subsets T_1 and T_2 of $\mathbb{R}$,
we define $T_1 + T_2 := \{t_1 + t_2 | t_1 \in T_1, t_2 \in T_2\}$.
Let $S_1 := (\mathbb{Q} \cap [0,1]) + \{\sqrt{2}\}$ and
$S_2 := (\mathbb{Q} \cap [1,2]) + \{\sqrt{3}\}$. Let $S = S_1 \cup S_2$.
Which one of the following is true?
(A) There exists a one one map from [0,1] to S
(B) There exists an onto map from S to [0,1]
(C) $S_1 = S_2$
(D) None of the above

11. Let X, Y and Z be finite sets and let $f: X\ \to Y$ and $g : Y \to Z$ be maps.
Which of the following assertions is always true?
(A) If gof is a bijection, then both g and f are bijections.
(B) If g is one to one, then gof is also one to one.
(C) If f is onto, then gof is also onto.
(D) If gof is onto, then $|Z| \leq |Y|$, where $|A|$ denotes the number of elements in any finite set A.