GRADUATE SCHOOL ADMISSIONS (GS)

TIFR MATHEMATICS

Topic & Subtopic-Wise Sorted

PYQs

with Detailed Explanation

16 Years

GS2010 To GS2025

Useful for
IIT JAM, GAT-B, CUET (PG), GATE, Autonomous, Deemed,
Private, State & Central Universities Entrance Exams

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P R E F A C E

The principal purpose of writing this book **"TIFR MATHEMATICS"** is to cater to the needs of the candidates aspiring for **"TIFR GS - MATHEMATICS"** exam. It provides explanations of the previous year questions from the year **2010 to 2025**. To help students, we have compiled this book in a user-friendly and well-structured manner. Book includes seven chapters according to the main topics and each chapter is further subdivided into units. This fine sorting of questions will help the students to prepare for each subtopic effectively.

The book will help the reader to understand and solve all the previous year questions. Regular practice is the key to success in any competitive exam, and this book allows you to hone your problem-solving skills. We have tried to keep the material as simple as possible with easy explanations. The practice of these questions will help the students not only for TIFR exam but also for other competitive exams like CSIR-NET, GATE, PhD entrances, etc.

A word or suggestion from your side may add another feather to the cap in the compilation of questions in this book. The author looks forward to the comments, suggestions, and criticism from the students. Constructive suggestions and user feedback would be highly appreciated, gratefully acknowledged, and suitably incorporated. We wish all the best to all aspirants who are preparing for these exams.

SYLLABUS

Algebra: Definitions and examples of groups (finite and infinite, commutative and non-commutative), cyclic groups, subgroups, homomorphisms, quotients. Group actions and Sylow theorems. Definitions and examples of rings and fields. Integers, polynomial rings and their basic properties. Basic facts about vector spaces, matrices, determinants, ranks of linear transformations, characteristic and minimal polynomials, symmetric matrices. Inner products, positive definiteness.

Analysis: Basic facts about real and complex numbers, convergence of sequences and series of real and complex numbers, continuity, differentiability and Riemann integration of real valued functions defined on an interval (finite or infinite), elementary functions (polynomial functions, rational functions, exponential and log, trigonometric functions), sequences and series of functions and their different types of convergence.

Geometry/Topology: Elementary geometric properties of common shapes and figures in 2 and 3 dimensional Euclidean spaces (e.g. triangles, circles, discs, spheres, etc.). Plane analytic geometry (= coordinate geometry) and trigonometry. Definition and basic properties of metric spaces, examples of subset Euclidean spaces (of any dimension), connectedness, compactness. Convergence in metric spaces, continuity of functions between metric spaces.

General: Pigeon-hole principle (box principle), induction, elementary properties of divisibility, elementary combinatorics (permutations and combinations, binomial coefficients), elementary reasoning with graphs, elementary probability theory

EXAM PATTERN

The paper is divided into two sections - Part A and Part B. Part A consists of 20 multiple choice questions. Each question from this part has four options, of which exactly one is correct. Part B consists of 20 true/false questions.

Grading Scheme: There will be negative marking for wrong answers. The grading scheme is as shown in the following table:

	Correct Answer	Wrong Answer	No Answer
Part A	+ 2	- 1	0
Part B	+ 2	- 2	0

I N D E X

CHAPTER NO.	CHAPTER NAME	PAGE NO.
1	**REAL ANALYSIS**	**1**
	Unit 1.1: Functions & their properties	**1**
	Unit 1.2: Point set topology and countability	**9**
	Unit 1.3: Sequence of Real Numbers	**12**
	Unit 1.4: Series of Real Numbers	**18**
	Unit 1.5: Limits and Continuity of a Function	**22**
	Unit 1.6: Differentiability and its applications	**32**
	Unit 1.7: Riemann Integration and Functions of Bounded Variation	**38**
	Unit 1.8: Sequence and Series of Functions	**40**
	Unit 1.9: Functions of Several Variables	**46**
	Unit 1.10: Measure Theory	**48**
2	**LINEAR ALGEBRA**	**51**
	Unit 2.1: Matrices and their properties	**51**
	Unit 2.2: Vector Spaces, Dimension and Counting	**60**
	Unit 2.3: Linear Transformation and their Properties	**67**
	Unit 2.4: System of Linear Equations	**72**
	Unit 2.5: Eigenvalues and Eigenvectors	**73**
	Unit 2.6: Diagonalizability and Canonical Forms	**78**
	Unit 2.7: Inner product spaces	**80**
3	**COMPLEX ANALYSIS**	**82**
4	**ABSTRACT ALGEBRA**	**86**
	Unit 4.1: Number theory	**86**
	Unit 4.2: Groups and subgroups	**90**
	Unit 4.3: Cyclic Groups and Permutation Groups	**96**
	Unit 4.4: Normal subgroups, factor groups, class equation	**99**
	Unit 4.5: Homomorphism	**102**
	Unit 4.6: Sylow Theorems and Applications	**106**
	Unit 4.7: Important structures of Rings	**109**
	Unit 4.8: Ideals and Homomorphism	**113**
	Unit 4.9: Rings of polynomials and irreducibility	**117**
	Unit 4.10: ED, PID, UFD	**120**
5	**ORDINARY DIFFERENTIAL EQUATIONS**	**122**
6	**METRIC SPACES AND TOPOLOGY**	**126**
	Unit 6.1: Metric spaces and properties	**126**
	Unit 6.2: Topological spaces and properties	**132**
	Unit 6.3: Continuity, Connectedness and Compactness	**133**
	Unit 6.4: Homeomorphism	**141**
7	**GENERAL**	**143**

TIFR MATHEMATICS: NUMBER OF QUESTIONS ASKED IN TIFR EXAM (2010 to 2025)

CHAPTER: 1- REAL ANALYSIS		2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
1	Functions & their properties	2	4	1	4	6	2	1	2	5	2	1	2	2	-	1	2
2	Point set topology and countability	1	1	-	1	-	-	2	-	1	-	-	1	1	-	-	3
3	Sequence of Real Numbers	2	1	-	4	1	1	-	1	1	2	2	1	-	3	-	2
4	Series of Real Numbers	2	1	1	1	-	2	5	-	-	-	-	1	1	-	-	-
5	Limits and Continuity of a Function	4	6	3	4	1	3	5	6	5	3	6	2	3	3	-	2
6	Differentiability and its applications	1	1	3	-	2	4	2	2	1	2	3	-	-	-	-	1
7	Riemann Integration and Functions of Bounded Variation	-	1	2	-	-	-	-	-	-	-	-	-	-	-	-	-
8	Sequence and Series of Functions	1	1	1	-	1	-	2	1	-	3	-	2	2	1	1	2
9	Functions of Several Variables	-	1	-	-	-	-	-	-	1	-	-	-	-	-	-	-
10	Measure Theory	-	1	-	-	-	-	-	-	-	-	-	-	-	-	-	-

REAL ANALYSIS

CHAPTER 1

Unit 1.1: Functions and their properties

MCQ (Multiple Choice Questions)

[TIFR 2010]

1. Let f be a one-to-one function from the closed interval [0, 1] to the set of real numbers R, then
   (a) f must be onto.
   (b) Range of f must contain a rational number.
   (c) Range of f must contain an irrational number.
   (d) Range of f must contain both rational and irrational numbers.

[TIFR 2010]

2. The sum of the roots of the equation $x^5 + 3x^2 + 7 = 0$ is
   (a) -3 (b) $\frac{3}{7}$
   (c) $\frac{-1}{7}$ (d) 0

[TIFR 2011]

3. Let $m \leq n$ be natural numbers. The number of injective maps from a set of cardinality m to a set of cardinality n is
   (a) $m!$ (b) $n!$
   (c) $(n - m)!$ (d) none of the above

[TIFR 2014]

4. Let C be set of integer pairs (a, b) for which the three complex roots r_1, r_2 and r_3 of the polynomial $p(x) = x^3 - 2x^2 + ax - b$ satisfy $r_1^3 + r_2^3 + r_3^3 = 0$. Then the cardinality of C is
   (a) $|C| = \infty$ (b) $|C| = 0$
   (c) $|C| = 1$ (d) $1 < |C| < \infty$

[TIFR 2014]

5. Let $f : [0, 1] \to R$ be a continuous function. Which of the following statements is always true?
   (a) $\int_0^1 f^2(x)dx = \left(\int_0^1 f(x)dx\right)^2$
   (b) $\int_0^1 f^2(x)dx \leq \left(\int_0^1 |f(x)|\, dx\right)^2$
   (c) $\int_0^1 f^2(x)dx \geq \left(\int_0^1 |f(x)|\, dx\right)^2$
   (d) $\int_0^1 f^2(x)dx \leq \left(\int_0^1 f(x)dx\right)^2$

[TIFR 2014]

6. Let $f : R \to R$ be a continuous function such that $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$, for all $x, y \in R$. Then
   (a) f is both one-to-one and onto
   (b) f is one-to-one but may not be onto
   (c) f is onto but may not be one-to-one
   (d) f is neither one-to-one nor onto.

[TIFR 2014]

7. There exists a map $f : Z \to Q$ such that f
   (a) is bijective and increasing
   (b) is onto and decreasing
   (c) is bijective and satisfies $f(n) \geq 0$ if $n \leq 0$
   (d) has uncountable image.

[TIFR 2014]

8. Let $f : [0, 1] \to [0, \infty)$ be continuous. Suppose $\int_0^x f(t)dt \geq f(x)$,for all $x \in [0, 1]$. Then
   (a) no such function exists
   (b) there are infinitely many such functions
   (c) there is only one such function
   (d) there are exactly two such functions.

[TIFR 2014]

9. Let $f : R^2 \to R$ be a continuous map such that $f(x) = 0$ for only finitely many values of x. Which of the following is true?
   (a) either $f(x) \leq 0$ for all x, or, $f(x) \geq 0$ for all x
   (b) the map f is onto
   (c) the map f is one-to-one
   (d) none of the above.

[TIFR 2015]

10. A complex number $\alpha \in \mathbb{C}$ is called $algebraic$ if there is a non-zero polynomial $P(x) \in \mathbb{Q}[x]$ with rational coefficients such that $P(\alpha) = 0$. Which of the following statements is true?
    (a) There are only finitely many algebraic numbers
    (b) All complex numbers are algebraic
    (c) $sin\left(\frac{\pi}{3}\right) + cos\left(\frac{\pi}{4}\right)$ is algebraic
    (d) None of the above.

[TIFR 2015]

11. Let f be a function from $\{1, 2, \ldots, 10\}$ to R such that $\left(\sum_{i=1}^{10} \frac{|f(i)|}{2^i}\right)^2 = \left(\sum_{i=1}^{10} |f(i)|^2\right)\left(\sum_{i=1}^{10} \frac{1}{4^i}\right)$. Mark the correct statement.
    (a) There are uncountably many f with this property
    (b) There are only countably infinitely many f with this property
    (c) There is exactly one such f
    (d) There is no such f.

[TIFR 2016]

12. Let $p(x)$ be a polynomial of degree 3 with real coefficients. Which of the following is possible?
(a) $p(x)$ has no real roots
(b) $p(x)$ has exactly 2 real roots
(c) $p(1) = -1, p(2) = 1, p(3) = 11$ and $p(4) = 35$
(d) $i - 1$ and $i + 1$ are roots of $p(x)$, where i is the square root of -1

[TIFR 2018]

13. How many zeros does the function $f(x) = e^x - 3x^2$ have in R?
(a) 0 (b) 1
(c) 2 (d) 3.

[TIFR 2018]

14. Suppose p is a degree 3 polynomial such that $p(0) = 1, p(1) = 2,$ and $p(2) = 5$. Which of the following numbers cannot equal $p(3)$?
(a) 0 (b) 2
(c) 6 (d) 10.

[TIFR 2018]

15. Let A be the set of all function $f : R \to R$ that satisfy the following two properties:
- f has derivatives of all order, and
- For all $x, y \in R, f(x + y) - f(y - x) = 2xf'(y)$.

Which of the following sentences is true?
(a) Any $f \in A$ is a polynomial of degree less than or equal to 1.
(b) Any $f \in A$ is a polynomial of degree less than or equal to 2.
(c) There exists $f \in A$ which is not a polynomial.
(d) There exists $f \in A$ which is a polynomial of degree 4.

[TIFR 2018]

16. Consider the following four sets of maps $f : \mathbb{Z} \to \mathbb{Q}$:
(i) $\{f : Z \to Q \mid f$ is bijective and increasing$\}$.
(ii) $\{f : Z \to Q \mid f$ is onto and increasing$\}$,
(iii)$\{f : Z \to Q \mid f$ is bijective, and satisfies that $\forall n \leq 0, f(n) \geq 0\}$, and
(iv)$\{f : Z \to Q \mid f$ is onto and decreasing$\}$.
How many of these sets are empty?
(a) 0 (b) 1
(c) 2 (d) 3.

[TIFR 2019]

17. Consider differentiable function $f: R \to R$ with the property that for all $a, b \in R$, we have: $f(b) - f(a) = (b - a)f'\left(\frac{a+b}{2}\right)$. Then which one of the following sentences is true?
(a) Every such f is a polynomial of degree less than or equal to 2
(b) There exists such a function f which is a polynomial of degree bigger than 2
(c) There exists such a function f which is not a polynomial
(d) Every such f satisfies the condition $f\left(\frac{a+b}{2}\right) \leq \frac{f(a)+f(b)}{2}$ for all $a, b \in R$

[TIFR 2019]

18. Let $C^\infty(0,1)$ stand for the set of all real-valued functions on (0,1) that have derivatives of all orders. Then the map $C^\infty(0,1) \to C^\infty(0,1)$ given by $f \mapsto f + \frac{df}{dx}$ is
(a) injective but not surjective
(b) surjective but not injective
(c) neither injective nor surjective
(a) both injective and surjective

[TIFR 2020]

19. Let $f: R \to R$ be a continuous function such that: $|f(x) - f(y)| \geq \log(1 + |x - y|)$, for all $x, y \in R$. Then:
(a) f is injective but not surjective
(b) f is surjective but not injective
(c) f is neither injective nor surjective
(d) f is bijective

[TIFR 2021]

20. What is the number of surjective maps from the set $\{1, \ldots, 10\}$ to the set $\{1, 2\}$?
(a) 90 (b) 1022
(c) 98 (d) 1024

[TIFR 2021]

21. The number of bijective maps $g : N \to N$ such that $\sum_{n=1}^{\infty} \frac{g(n)}{n^2} < \infty$ is
(a) 0 (b) 1
(c) 2 (d) ∞

[TIFR 2022]

22. What is the number of solutions of $x = \frac{x^2}{50} - \cos\frac{x}{2} + 2$ in [0, 10]?
(a) 0 (b) 1
(c) 2 (d) ∞

[TIFR 2022]

23. Consider the real polynomial
$f(x) = x^{11} - x^7 + x^2 - 1.$
Which of the following sentences is correct?
(a) $f(x)$ has exactly one positive root.
(b) $f(x)$ has exactly two positive roots.
(c) $f(x)$ has at least three positive roots.
(d) None of the other three options.

[TIFR 2024]

24. Let n be a positive integer, and let $S = \{g \in R[x] \mid g$ is a polynomial of degree at most $n\}$.
For $g \in S$, let $A_g = \{x \in R | e^x = g(x)\} \subset R$. Let $m = \min\{\#\, A_g | g \in S\}$, and $M = \max\{\#\, A_g | g \in S\}$
Then
(a) $m = 0, M = n$ (b) $m = 0, M = n + 1$
(c) $m = 1, M = n$ (d) $m = 1, M = n + 1$

[TIFR 2025]

25. What is the number of functions $f: \mathbb{R} \setminus \{0,1\} \to \mathbb{R} \setminus \{0,1\}$ such that $|f(x) - f(y)| = |x - y|$ for all $x, y \in \mathbb{R} \setminus \{0,1\}$?
    (a) 1
    (b) 2
    (c) Greater than 2, but finite
    (d) Infinite

True/False

[TIFR 2011]

1. $e^{\sqrt{2}} > 3$.

[TIFR 2011]

2. For any real number c, the polynomial $x^3 + x + c$ has exactly one real root.

[TIFR 2011]

3. The polynomial $x^4 + 7x^3 - 13x^2 + 11x$ has exactly one real root.

[TIFR 2012]

4. The inequality $\sqrt{1+x} < 1 + \frac{x}{2}$ for $x \in (-1,10)$ is true

[TIFR 2013]

5. There exists a bijection between R^2 and the open interval $(0,1)$.

[TIFR 2013]

6. The equation $x^3 + 3x - 4 = 0$ has exactly one real root.

[TIFR 2013]

7. The equation $x^3 + 10x^2 - 100x + 1729 = 0$ has at least one complex root α such that $|\alpha| > 12$.

[TIFR 2013]

8. The function $f : Z \to R$ defined by $f(n) = n^3 - 3n$ is injective.

[TIFR 2017]

9. There exists a function $f: R \to R$ satisfying, $f(-1) = -1$, $f(1) = 1$ and $|f(x) - f(y)| \leq |x - y|^{\frac{3}{2}}$, for all $x, y \in R$

[TIFR 2017]

10. Let P be a monic, non-zero, polynomial of even degree, and $K > 0$. Then the function $P(x) - Ke^x$ has a real zero.

[TIFR 2018]

11. Let A be a countable subset of R which is well-ordered with respect to the usual ordering on R (where 'well-ordered' means that every nonempty subset has a minimum element in it). Then A has an order preserving bijection with a subset of N.

[TIFR 2025]

12. There exists a linear polynomial $p(x) \in \mathbb{R}[x]$, for which there exist exactly two subsets $S \subset \mathbb{R}$ such that $p(S) = S$ and such that $\#S = 2$.

Answer Key

Multiple Choice Questions (MCQs)									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
c	d	d	b	c	a	c	c	a	c
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
a	c	d	d	b	d	a	b	d	b
21	**22**	**23**	**24**	**25**					
a	b	a	b	b					

True or False									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
True	True	False	False	True	True	True	False	False	True
11	**12**								
False	False								

EXPLANATION

Multiple Choice Questions (MCQs)

1. EXPLANATION:

$f(x) =$ x discards (a).

$f(x) = \begin{cases} x & x \in Q^c \cap [0,1] \\ \sqrt{3}+x & x \in Q \cap [0,1] \end{cases}$

Note that f is injective but $R(f)$ does not contain rational number. This discard (b) and (d).

As $[0,1]$ is uncountable and f is injective, $f([0,1])$ is also uncountable, so it must contain an irrational number.

2. EXPLANATION:

We know that for

$f(x) = a_n x^n + a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + \cdots + a_0$, sum of roots is $-\frac{a_{n-1}}{a_n}$

$\therefore \frac{0}{1} = 0$ is the sum of roots.

3. EXPLANATION:

The number of injective maps from a set of cardinality m to a set of cardinality n is $n_{P_m} = \frac{n!}{(n-m)!}$

4. EXPLANATION:

$p(x) = x^3 - 2x^2 + ax - b = (x - r_1)(x - r_2)(x - r_3)$

Then we have

$r_1 + r_2 + r_3 = 2$

$r_1r_2 + r_2r_3 + r_3r_1 = a$

$r_1r_2r_3 = b$

$r_1^3 + r_2^3 + r_3^3 = [(r_1 + r_2 + r_3)(r_1^2 + r_2^2 + r_3^2 - r_1r_2 - r_2r_3 - r_1r_3) + 3r_1r_2r_3]$

$\Rightarrow 0 = (r_1 + r_2 + r_3)[(r_1 + r_2 + r_3)^2 - 3(r_1r_2 + r_2r_3 + r_1r_3)] + 3r_1r_2r_3$

$\Rightarrow 0 = 2(4 - 3\text{a}) + 3\text{b}$

$\Rightarrow 0 = 8 - 6\text{a} + 3\text{b}$

$\Rightarrow 6a - 3b = 8$

$\Rightarrow 3a - b = \frac{8}{3}$

But if a and b both are integers, then so is $3a - b$. But $3a - b = \frac{8}{3}$, which is absurd.

5. EXPLANATION:

Cauchy-Schwarz inequality for integrals says

$\int_0^1 h^2 \int_0^1 g^2 \geqslant \left(\int_0^1 hg\right)^2$

Take $h = |f|$ and $g = 1$, to get (c).

6. EXPLANATION:

$|f(x) - f(y)| \geqslant \frac{1}{2}|x - y|$

$\Rightarrow \frac{|f(x)-f(y)|}{|x-y|} \geqslant \frac{1}{2}$

$\Rightarrow \lim_{x \to y} \frac{|f(x)-f(y)|}{|x-y|} \geqslant \frac{1}{2}$

$\Rightarrow f'(x) \geqslant \frac{1}{2}, \forall x$

This implies f is strictly increasing, hence one to one.

Also given that $|f(x) - f(y)| \geq \frac{1}{2}|x - y|$.

Take $y = 0$. Then $|f(x) - f(0)| \geq \frac{1}{2}|x|$.

Therefore, $-|f(x) - f(0)| \leq \frac{x}{2} \leq |f(x) - f(0)|$.

So, we have $|f(x)| \geq \frac{x}{2} - |f(0)|$ and $|f(x)| \leq \frac{x}{2} + |f(0)|$, which implies f is not bounded above and below.

Hence, f is onto.

7. EXPLANATION:

Note that $\mathbb{Z}^- \cup \{0\}$ is countable set and $\mathbb{Q}^+ \cup \{0\}$ is also countable.

Therefore, there exists bijection, say f_1 between these two sets.

Similarly, $\mathbb{Z}^+$ and $\mathbb{Q}^-$ are countable and thus have bijection, say f_2 between them.

Concating f_1 and f_2 gives required map.

8. EXPLANATION:

Note that $\frac{d}{dx}\left(e^{-x}\int_0^x f(t)dt\right) = e^{-x}\left(f(x) - \int_0^x f(t)dt\right) \leq 0$

So, $e^{-x}\int_0^x f(t)dt$ is non-increasing and takes value 0 at $x = 0$.

$\therefore e^{-x}\int_0^x f(t)dt \leq 0$ implies $\int_0^x f(t)dt \leq 0$

Therefore, we have $0 \leq f(x) \leq \int_0^x f(t)dt \leq 0$ which gives $f \equiv 0$.

9. EXPLANATION:

If f changes sign, say $f(A) < 0$ and $f(B) > 0$, then for any path ϕ_{AB} from A to B, function $f \circ \phi_{AB}$ has a zero, which means f has a zero. As, there are infinitely many paths, this will lead to infinitely many zeros of f, a contradiction.

10. EXPLANATION:

We know that set of all algebraic numbers is countable. This discard (a) and (b).

Now, $\sin\left(\frac{\pi}{3}\right) + \cos\left(\frac{\pi}{4}\right) = \frac{\sqrt{3}}{2} + \frac{1}{\sqrt{2}} = \frac{\sqrt{3}+\sqrt{2}}{2}$

Let $x = \frac{\sqrt{3}+\sqrt{2}}{2}$

Then $2x = \sqrt{3} + \sqrt{2}$

$\therefore 4x^2 = 5 + 2\sqrt{6}$

$\therefore (4x^2 - 5)^2 = 24$

So, x is solution of $(4x^2 - 5)^2 - 24 \in \mathbb{Q}[x]$.

Therefore, x is algebraic.

11. EXPLANATION:

If we take $f(i) = k2^i$, for some scalar k, we have equality by Cauchy-Schwarz.

As, there are uncountably many choices for k, (a) is correct.

12. EXPLANATION:

odd degree real polynomial has at least one real root, so option (a) incorrect.

$p(x) = x^3 - 1$ has only one real root, so (b) is incorrect.

As complex roots are present in conjugate pair, (d) is wrong.

13. EXPLANATION:

It is easy to see that in interval $[-2, 2]$, graphs of these two will intersect in two points.

Even though graph of x^3 dominates e^x for some part, e^x grows faster than $3x^2$. This guarantees that there will be one more intersecting point, where e^x starts dominating $3x^2$.

14. EXPLANATION:

Let $p(x) = ax^3 + bx^2 + cx + d$

$p(0) = 1 \Rightarrow \text{d} = 1$ ... (1)

$p(1) = 2 \Rightarrow a + b + c + d = 2 \Rightarrow \text{a} + \text{b} + \text{c} = 1$ ... (2)

$p(2) = 5 \Longrightarrow 4a + 2b + c = 2$ ... (3)

Subtract (2) from (3)

$\Longrightarrow 3a + b = 1 \Rightarrow b = 1 - 3a$

From (2), $\therefore c = 1 - a - b = 1 - a - 1 + 3a = 2a$

$p(3) = 27a + 9(1 - 3a) + 3(2a) + 1 = 27a + 9 - 27a + 6\text{a} + 1$

$\Rightarrow p(3) = 10 + 6a$

If $p(x) = 10$, then $a = 0$, contradiction to p is degree 3 polynomial

$\Rightarrow p(3) \neq 10$

15. EXPLANATION:

$f(x + y) - f(y - x) = 2xf'(y)$

Differentiate w.r.t x.

$f'(x + y) + f'(y - x) = 2f'(y)$

Again differentiating,

$f''(x + y) - f''(y - x) = 0, \forall x, y$

$\Rightarrow f''(x)$ is constant

$\Rightarrow f(x)$ is a polynomial of degree less that or equal to 2.

16. EXPLANATION:

(iii) is not empty because note that $\mathbb{Z}^- \cup \{0\}$ is countable set and $\mathbb{Q}^+ \cup \{0\}$ is also countable.

Therefore, there exists bijection, say f_1 between these two sets.

Similarly, $\mathbb{Z}^+$ and $\mathbb{Q}^-$ are countable and thus have bijection, say f_2 between them.

Concating f_1 and f_2 gives required map.

Other three sets are empty. Because topology of $\mathbb{Z}$ and $\mathbb{Q}$ is different. There is no integer between integers a and b such that $a < b$ but there are infinitely many rational numbers between $f(a)$ and $f(b)$.

17. EXPLANATION:

Put $b = x + y$ and $a = x - y$ In given equation, we get,

$\frac{f(b) - f(a)}{b - a} = f'\left(\frac{a+b}{2}\right)$

$f(x + y) - f(x - y) = 2yf'(x)$

Differentiating two times with respect to y, we get,

$f''(x + y) - f''(x - y) = 0, \forall x, y$

This implies $f''(x) = 0, \forall x$, which after integrating two times implies (a) is correct.

18. EXPLANATION:

Let T be the map.

Note that $T(e^{-x}) = T(2e^{-x}) = 0$ implies T is not one-one.

For any f, there exists $g(x) = e^{-x}\int e^x f$ such that $T(g) = f$, so T is onto.

Here $g'(x) = -e^{-x}\int e^x f + e^{-x}(e^x f) = f(x) - e^{-x}\int e^x f$.

19. EXPLANATION:

Given that $f: \mathbb{R} \to \mathbb{R}$ is a contimous function such that $|f(x) - f(y)| \geq \log(1 + |x - y|) \forall x, y \in \mathbb{R}$ If $x \neq y$, then $|f(x) - f(y)| \geq \log(1 + |x - y|) > 0$, which implies $f(x) \neq f(y)$, so f is injective.

Put $x = y + h$, we have

$\left|\frac{f(y+h) - f(y)}{h}\right| \geq \frac{\log(1+h)}{h} \Rightarrow \lim_{h \to 0} \left|\frac{f(y+h) - f(y)}{h}\right| \geq \lim_{h \to 0} \frac{\log(1+h)}{h}$

$\Rightarrow |f(y)| \geq 1$

$\Rightarrow f$ is either strictly increasing or decreasing function, which gives f is onto.

20. EXPLANATION:

Total function possible are 2^{10}.

Of which only two are not onto, one which maps everything to 1 and other which maps everything to 2.

Therefore, total onto maps are $2^{10} - 2$.

OR

le $A = \{1,2, \ldots, 10\}$ and $B = \{1,2\}$

$\therefore \text{n}(A) = 10$ and $n(B) = 2$

$\Rightarrow$ The number of onto (surjective) functions from the set A to set B are

$= 2^{10} - {}^2C_1(2 - 1)^{10} + {}^3C_2(2 - 2)^{10}$

$= 2^{10} - 2$

$= 1024 - 2 = 1022$

Here, we have used that if $n(A) = r$ and $n(B) = m$, then number of onto functions are

$m^r - {}^nC_1(m-1)^r + {}^mC_2(m-2)^r - {}^mC_3(m-3) + \cdots$

21. EXPLANATION:

Note that the sum $\sum_{n=1}^{\infty} \frac{g(n)}{n^2}$ is minimum when $g(n) = n$.

So, for any bijective map, $\sum_{n=1}^{\infty} \frac{g(n)}{n^2} \geq \sum_{n=1}^{\infty} \frac{n}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n}$, which gives series divergent.

22. EXPLANATION:

Let $f(x) = x - \frac{x^2}{50} + \cos\frac{x}{2} - 2$

Note that $f(0) = -1 < 0$ and $f(10) > 0$.

Using IVP, at least one root exists in $[0,10]$.

Also note that in $[0,10]$, $f'(x) > 0$, so maximum one root exists.

23. EXPLANATION:

$f(x) = x^{11} - x^7 + x^2 - 1 = x^7(x^4 - 1) + (x^2 - 1) =$
$x^7(x^2 - 1)(x^2 + 1) + (x^2 - 1)$
$\therefore f(x) = (x^2 - 1)(x^7(x^2 + 1) + 1)$
$f(x) = 0$
$\Rightarrow (x^2 - 1)(x^7(x^2 + 1) + 1) = 0$
$\Rightarrow (x^2 - 1) = 0$ or $x^7(x^2 + 1) + 1 = 0$

For $x^2 - 1 = 0$, we have one positive solution $x = 1$.

For $x^7(x^2 + 1) + 1 = 0$, there is no positive solution as for any $x > 0$, $x^7(x^2 + 1) > 0$.

24. EXPLANATION:

For $n = 1$, $g = -1$ gives $A_g = 0$ and $g = 3x$ gives $A_g = 2$. So, we have (b) correct.

True or false

25. EXPLANATION:

Any map preserving distances on a subset of $\mathbb{R}$ is the restriction of an affine isometry $f(x) = \varepsilon x + c$ with $\varepsilon \in \{1, -1\}$.

If $\varepsilon = 1$, then $f(x) = x + c$ must avoid 0,1 for every $x \neq 0,1$, which forces $c = 0$ (identity).

If $\varepsilon = -1$, then $f(x) = -x + c$ must avoid 0,1 for all $x \neq 0,1$, which forces $c = 1$ (the map $x \mapsto 1 - x$).

Thus, exactly two such functions exist.

True/False

1. EXPLANATION:

$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$
$e^{\sqrt{2}} = 1 + \sqrt{2} + \frac{(\sqrt{2})^2}{2!} + \frac{(\sqrt{2})^3}{3!} + \cdots$
$= 1 + 1.414.. + 1 + \frac{(\sqrt{2})^3}{3!} + \cdots$
$= 3 \cdot 414.. + \frac{(\sqrt{2})^3}{3!} + \cdots > 3$

2. EXPLANATION:

$f'(x) = 3x^2 + 1 > 0$
$\Longrightarrow f(x)$ is increasing
$\Longrightarrow f(x)$ has at most one real root
Here $f(x)$ is cubic polynomial
$\Longrightarrow f(x)$ has at least one real root

3. EXPLANATION:

Here $x = 0$ is one real root and other factor $x^3 + 7x^2 - 13x + 11$ will also have a non-zero real root.

4. EXPLANATION:

Take $x = 0$ to contradict.

5. EXPLANATION: Between any two sets with same cardinality, exists a bijection. Both the given sets have cardinality c, so bijection exists.

6. EXPLANATION:

$f(x)$ is polynomial of odd degree, so at least one real root is there.

$f'(x) = 3x^2 + 3 > 0$, so f is increasing.

So, f has exactly one root.

7. EXPLANATION:

If z_1, z_2, z_3 are the three roots, then $z_1 z_2 z_3 = -1729$, which gives $|z_1||z_2||z_3| = 1729$ and forces at least one root to have magnitude greater than 12 as $12^3 = 1728$.

8. EXPLANATION:

$f(1) = 1 - 3 = -2$
$f(-2) = -8 + 6 = -2$
$\Rightarrow f(1) = f(-2)$, which means f is not injective.

9. EXPLANATION:

$|f(x) - f(y)| \leqslant |x - y|^{3/2}$
$\therefore \frac{|f(x) - f(y)|}{|x - y|} \leqslant |x - y|^{1/2}$

Take $y \to x$, we get,

$|f'(x)| \leq 0$
$\Rightarrow f'(x) = 0$
$\Rightarrow f$ is constant, So, no such f exists.

Unit 1.8: Sequence and Series of Functions

MCQ (Multiple Choice Question)

[TIFR 2010]

1. If $f_n(x)$ are continuous functions from $[0,1]$ to $[0,1]$, and $f_n(x) \to f(x)$ as $n \to \infty$, then which of the following statements is true?
   (a) $f_n(x)$ converges to $f(x)$ uniformly on [0, 1]
   (b) $f_n(x)$ converges to $f(x)$ uniformly on $(0,1)$
   (c) $f(x)$ is continuous on $[0,1]$
   (d) None of the above

[TIFR 2011]

2. The function $f_n\,(x) = n\,sin\left(\frac{x}{n}\right)$
   (a) does not converge for any $x\ as\ n \to \infty$.
   (b) Converges to the constant function 1 as $n \to \infty$
   (c) converges to the function x as $\to \infty$.
   (d) does not converge for all x as $n \to \infty$.

[TIFR 2014]

3. Let $f_n(x)$, for $n \geq 1$, be a sequence of continuous nonnegative functions on $[0,1]$ such that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. Which of the following statements is always correct?
   (a) $f_n \to 0$ uniformly on $[0,1]$
   (b) f_n many not converge uniformly but converges to 0 point-wise
   (c) f_n will converge point-wise and the limit may be non-zero
   (d) f_n is not guaranteed to have a point-wise limit.

[TIFR 2016]

4. Let $\{f_n\}_{n=1}^{\infty}$ be the sequence of functions on R defined by $f_n(x) = n^2x^n$. Let A be the set of all points a in R such that the sequence $\{f_n(a)\}_{n=1}^{\infty}$ converges. Then
   (a) $A = \{0\}$ (b) $A = [0,1)$
   (c) $A = R \setminus \{-1,1\}$ (d) $A = (-1,1)$

[TIFR 2016]

5. Which of the following sequences of functions $\{f_n\}_{n=1}^{\infty}$ converges uniformly?
   (a) $f_n(x) = x^n$ on $[0,1]$
   (b) $f_n(x) = 1 - x^n$ on $\left[\frac{1}{2}, 1\right]$
   (c) $f_n(x) = \frac{1}{1+nx^2}$ on $\left[0, \frac{1}{2}\right]$
   (d) $f_n(x) = \frac{1}{1+nx^2}$ on $\left[\frac{1}{2}, 1\right]$.

[TIFR 2019]

6. Let f be a continuous function on $[0,1]$. Then the limit $lim_{n\to\infty} \int_0^1 nx^n f(x)dx$ is equal to
   (a) $f(0)$
   (b) $f(1)$
   (c) $f(x)$
   (d) The limit need not exist

[TIFR 2019]

7. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of function from $R\ to\ R$ definite by $f_n(x) = \frac{1}{2}exp(-n^2x^2)$ Then Which one of the following statements is true?
   (a) Both sequence $\{f_n\}$ and $\{f'_n\}$ converges uniformly on R
   (b) Neither $\{f_n\}$ nor $\{f'_n\}$ converges uniformly on R
   (c) $\{f_n\}$ converges point wise but not uniformly on any interval containing the origin
   (d) $\{f'_n\}$ converges point wise but not uniformly on any interval containing the origin

[TIFR 2019]

8. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of function from $R\ to\ R$ defined by $f_n(x) = \frac{\sqrt{1+(nx)^2}}{n}$. Then which one of the following statements is true?
   (a) $\{f_n\}$ and $\{f'_n\}$ converge uniformly on R
   (b) $\{f'_n\}$ converges uniformly on R but $\{f_n\}$ does not
   (c) $\{f_n\}$ converges uniformly on R but $\{f'_n\}$ does not
   (d) $\{f_n\}$ converges uniformly to a differentiable function on R

[TIFR 2021]

9. Let $X \subseteq \mathbb{R}$ be a subset. Let $\{f_n\}_{n=1}^{\infty}$ be a sequence of functions $f_n : X \to \mathbb{R}$, that converges uniformly to a function $f : X \to \mathbb{R}$. For each positive integer n, let $D_n \subseteq X$ denote the set of points at which f is not continuous. Which one of the following statements is correct?
   (a) If each D_n is finite, then D is finite.
   (b) If each D_n has at most 7 elements, then D has at most 7 elements.
   (c) If each D_n is uncountable, then D is uncountable
   (d) None of the other three statements is correct.

[TIFR 2022]

10. The value of $\int_0^1 \frac{ne^x}{1+n^2x^2}dx$ Is
    (a) πe (b) $\frac{\pi}{2}$
    (c) $\frac{\pi}{2}e$ (d) π

[TIFR 2023]

11. For every positive integer n, define $f_n : [0,1] \to R$ by $f_n(x) = \frac{sin\,(n^2x) + \ cos\,(e^nx)}{1+n^2x^2}$. Then $\int_0^{1-sin\,(1/n)} f_n(x)\,dx$ equals
    (a) 1. (b) 0.
    (c) ∞. (d) 1/2.

[TIFR 2024]

12. Let $f: [0,\infty) \to R$ be a continuous function. If $\int_0^1 f\,(x+n)\,dx = 2$, then which of the following statements about the limit $\int_0^1 f\,(nx)\,dx$ is correct?

(a) The limit exists and equals 0
(b) The limit exists and equals $\frac{1}{2}$
(c) The limit exists and equals 2
(d) None of the remaining three options is correct

[TIFR 2025]

13. For a positive integer n, define the function $f_n: \mathbb{R} \to \mathbb{R}$ by $f_n(x) = \sin\left(\frac{x}{n}\right)$. Consider the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f_n'\}_{n=1}^{\infty}$ of functions from $\mathbb{R}$ to $\mathbb{R}$. Which of the following statements is correct?
(a) The sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f_n'\}_{n=1}^{\infty}$ both converge uniformly on $\mathbb{R}$.
(b) Neither of the sequences $\{f_n\}_{n=1}^{\infty}$ and $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$.
(c) The sequence $\{f_n\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$, but the sequence $\{f_n'\}_{n=1}^{\infty}$ does not.
(d) The sequence $\{f_n'\}_{n=1}^{\infty}$ converges uniformly on $\mathbb{R}$, but the sequence $\{f_n\}_{n=1}^{\infty}$ does not.

[TIFR 2025]

14. Let S be the set of functions $f: (0,1) \to \mathbb{R}$ with the property that there is a sequence $\{f_n\}_{n=1}^{\infty}$ of functions that converges uniformly to f, where each $f_n: (0,1) \to \mathbb{R}$ is a twice continuously differentiable function (i.e., f_n'' exists and is continuous). Which of the following statements is correct?
(a) Any $f \in S$ is continuous, but there exists $f \in S$ such that f is not differentiable.
(b) Any $f \in S$ is differentiable, but there exists $f \in S$ such that f' is not continuous.
(c) Any $f \in S$ is continuously differentiable, but there exists $f \in S$ such that f'' does not exist.
(d) Any $f \in S$ is twice continuously differentiable.

True or False

1. Suppose $f_n(x)$ is a sequence of continuous functions on the closed interval $[0, 1]$ converging to 0 pointwise. Then the integral $\int_0^1 f_n(x)dx$ converges to 0.

[TIFR 2012]

2. If f is continuous on $(0,1)$and if $\int_0^1 f(x)x^n\, dx = 0$ for $n = 0,1,2,3,\ldots$. Then$\int_0^1 f^2(x)dx = 0$.

[TIFR 2017]

3. There exists a non-negative continuous function $f: [0, 1] \to R$ such that $\int_0^1 f^n dx \to 2$ as $n \to \infty$.

[TIFR 2021]

4. Let $f_n[0, 1] \to R$ be a continuous function for each positive integer n. If $\int_0^1 f_n\,(x)^2\, dx = 0$, then $f_n\left(\frac{1}{2}\right) = 0$.

[TIFR 2022]

5. There exists $f \in C([0, 1], R)$ satisfying the following two conditions:
(i) $\int_0^1 f(x)dx = 1$; and
(ii) $lim_{n\to\infty} \int_0^1 f(x)^n\ dx = 0$.

Answer Key

Multiple Choice Questions (MCQs)									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
a	d	d	d	d	b	d	c	b	b
11	**12**	**13**	**14**						
b	c	d	a						

True or False				
1	**2**	**3**	**4**	**5**
True	True	False	False	False

EXPLANATIONS

MCQ (Multiple Choice Question)

1. EXPLANATION:

Consider $f_n(x) = x^n$.

Then f_n converges pointwise to $f(x) = \begin{cases} 0, & 0 \le x < 1 \\ 1, & x = 1 \end{cases}$.

Clearly, $f(x)$ is not continuous.

Also, the convergence is not uniform as each f_n is continuous but f is not.

Also, the convergence is not uniform on $(0,1)$.

To see this, note that $f_n\left(\sqrt[n]{\frac{1}{2}}\right) = \frac{1}{2}$, so $M_n = ||f_n - f||_\infty \ge \frac{1}{2}$.

2. EXPLANATION:

$< f_n(x) \ge n \sin\left(\frac{x}{n}\right)$

$\lim_{n\to\infty} f(n) =$

$\lim_{n\to\infty} n \sin\left(\frac{x}{n}\right)$ …… $(\infty.0\ form)$

$\lim_{n\to\infty} f(n) \lim_{n\to\infty} \frac{\sin\left(\frac{x}{n}\right)}{\left(\frac{1}{n}\right)}$

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \frac{\sin\left(\frac{x}{n}\right)\left(\frac{-1}{n^2}\right)}{\frac{-1}{n^2}}$ $(By\ L - Hospital\ )$

$\lim_{n\to\infty} f(n) = \lim_{n\to\infty} \sin\left(\frac{x}{n}\right)$

$\lim_{n\to\infty} f_n(x) = 0$

3. EXPLANATION:

Consider

$f_n(x) = \begin{cases} x^n & \text{if } n \text{ is even} \\ 0 & \text{otherwise} \end{cases}$

Then $\int_0^1 f_{2n}(x)dx = \int_0^1 x^{2n} \text{dx} = \left[\frac{x^{2n+1}}{2n+1}\right]_0^1 = \frac{1}{2n+1} \to 0$ as $n \to \infty$

But $\langle f_n(1)\rangle = \langle 0,1,0,1,\cdots\rangle$ which is not convergent.

So, pointwise limit does not exist.

4. EXPLANATION:

For $x \ge 1$, n^2x^n diverges to ∞ and for $x \le -1$, n^2x^n is oscillating and unbounded.

For $-1 < x < 1$, $n^2x^n = \frac{n^2}{y^n}$, for $|y| > 1$.

It is of $\frac{\infty}{\infty}$ form, so by L'Hospital's rule, it converges to 0.

5. EXPLANATION:

(a) $f_n(x) = x^n$ on $[0,1]$

$\lim_{x\to\infty} x^n = \begin{cases} 0 & 0 \leqslant x < 1 \\ 1 & x = 1 \end{cases}$

Sequence of continuous functions converges pointwise to discontinuous function.

⇒ convergence is not uniform.

(b) $f_n(x) = 1 - x^n;\ x \in \left[\frac{1}{2}, 1\right]$

$\lim_{x\to\infty} f_n(x) = \begin{cases} 1 & 1/2 \leqslant x < 1 \\ 0 & x = 1 \end{cases}$

Sequence of continuous functions converges pointwise to discontinuous function.

⇒ convergence is not uniform.

(c) $f_n(x) = \frac{1}{1+nx^2};\ x \in \left[0, \frac{1}{2}\right]$

$\lim_{x\to\infty} f_n(x) = \begin{cases} 0 & 0 < x \le 1/2 \\ 1 & x = 0 \end{cases}$

Sequence of continuous functions converges pointwise to discontinuous function.

⇒ convergence is not uniform.

(d) Here $f(x) \equiv 0$.

Note that for $x \in \left[\frac{1}{2}, 1\right]$, $x^2 \in \left[\frac{1}{4}, 1\right]$ and thus $\frac{1}{x^2} \le 4$.

Therefore $\left|\frac{1}{1+nx^2}\right| \le \left|\frac{1}{nx^2}\right| \le \frac{4}{n}$ and thus $M_n = ||f_n - f||_\infty \le \frac{4}{n}$ tends to 0 as n tends to infinity.

So, convergence is uniform.

6. EXPLANATION:

Let F be a continuous function on [0,1]

If F is a continuous real valued function on [a, b]

Then we can uniformly approximate it by a polynomial to any degree of accuracy on [a, b]

Let $\lim_{n\to\infty} \int_0^1 fn(x)x^n \text{ndx} = \lim_{n\to\infty} \int_0^1 nx^n \sum a_k x^k$ dx

$= \lim_{n-\infty} \sum_{k=0}^m a_k \int_0^1 n\, x^{k+n}$

$= \lim_{n-x} \sum_{k=0}^m a_k \left[\frac{n}{n+k+1}\right]$

$= \sum_{k=0}^m a_k$ = f(1)

7. EXPLANATION:

$f_n(\text{x}) \equiv \frac{1}{n} e^{-n^2x^2}$

$\lim_{n\to\infty} f_n(x) = f(x) = 0$

M_n = SUP | Fn (x) – f(x) | = SUP $|\frac{1}{n} e^{-n^2x^2}\text{-}0|$

Let X = $\frac{1}{n}$

$\lim_{n-x} M_n = \lim_{n\to\infty} \frac{1}{n} = 0$

Also $f_n'(x) = \frac{-2nx}{\in n^2x^2}$

$\lim_{n\to\infty} f'n(x) = 0\ \forall\ \text{x} \in \mathbb{R}$

Mn' = sup $|f_n'(x) - 0|$ = SUP $|\frac{-2nx}{e^{n^2x^2}}| = \frac{2}{e} \longrightarrow 0$ ans $\to \infty$

$\Longrightarrow$ {fn' (x)} not converges uniformly

Option d correct

CHAPTER 3

COMPLEX ANALYSIS

Multiple Choice Questions (MCQs)

[TIFR 2020]

1. Let $\{z_1, z_2, \ldots, z_7\}$ be a set of seven complex numbers with unit modulus. Assume that they form the vertices of a regular heptagon in the complex plane. Define $\omega = \sum_{i<j} z_i z_j$. Then

   (a) $w = 0$ (b) $|w| = \sqrt{7}$

   (c) $|w| = 7$ (d) $|w| = 1$

[TIFR 2025]

2. The number of real cubic polynomials of the form $x^3 + ax + b$, each of whose complex roots lies on the unit circle $S^1 = \{z \in \mathbb{C} \mid z\bar{z} = 1\}$, equals

   (a) 0

   (b) 2

   (c) ∞

   (d) None of the remaining three options

True/False

[TIFR 2011]

1. Let P be a degree 3 polynomial with complex coefficients such that the constant term is 2010. Then P has a root a with $|a| > 10$.

[TIFR 2012]

2. There exist polynomials $f(x)$ and $g(x)$, with complex coefficients, such that $\left(\frac{f(x)}{g(x)}\right)^2 = x$.

[TIFR 2012]

3. If $z_1, z_2, z_3, z_4 \in \mathbb{C}$ satisfy $z_1 + z_2 + z_3 + z_4 = 0$ and $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = 1$, then the least value of $|z_1 - z_2|^2 + |z_1 - z_4|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2$ is 2.

[TIFR 2017]

4. Let α, β be complex numbers with non-positive real parts. Then $\left|e^{\alpha} - e^{\beta}\right| \leq |\alpha - \beta|$.

Answer Key

Multiple Choice Questions (MCQs)	
1	**2**
a	b

True or False			
1	**2**	**3**	**4**
False	False	True	True

EXPLANATION

Multiple Choice Questions (MCQs)

1. EXPLANATION:

As the vertices lie on a unit circle with centre at origin in complex plane, it can be argued that the vertices are all the roots of equation $z^7 - 1 = 0$.

Therefore, $z^7 - 1 = (z - z_1)(z - z_2)\cdots(z - z_7) = z^7 + (\sum z_i)z^6 - (\sum_{i<j}\ z_i z_j\ )z^5 + \cdots - \prod z_i$

So, by equating coefficients on both sides, we get $\sum_{i<j}\ z_i z_j = 0$.

2. EXPLANATION:

For $p(x) = x^3 + ax + b$ with real coefficients, if all roots lie on the unit circle, then their product has modulus 1, so $|b| = 1$.

Real roots on the unit circle can only be ± 1.

A real cubic with real coefficients either has three real roots or one real and a complex-conjugate pair.

Three real unit roots would be among $\{\pm 1\}$, but no triple of ± 1 sums to 0 (the coefficient of x^2 is $-$sum of roots and here it is 0), so the cubic must have one real root $r \in \{\pm 1\}$ and a conjugate pair $e^{\pm i\theta}$.

Sum of roots $= r + 2\cos\theta = 0$ gives $2\cos\theta = -r$.

For $r = 1$ we get $\cos\theta = -\frac{1}{2}$ (so $\theta = 2\pi/3$) and for $r = -1$ we get $\cos\theta = \frac{1}{2}$ (so $\theta = \pi/3$).

In each case the pair multiplies to 1, so the pairwise-sum term a equals $r \cdot 2\cos\theta + 1 = -r^2 + 1 = 0$.

Thus, the only possibilities are $x^3 - 1$ (when $r = 1$) and $x^3 + 1$ (when $r = -1$), and both have all roots on the unit circle.

Hence exactly two such polynomials.

For $p(x) = ax + b$ with $a \neq \pm 1$, the equation $p^2(x) = x$ is linear and has at most one solution, so there is no 2-cycle and hence no 2-element invariant set.

If $a = 1$, then $p(x) = x + b$; $b = 0$ gives the identity (every 2-point set is invariant, infinitely many), $b \neq 0$ gives no fixed 2-cycle.

If $a = -1$, then $p(x) = -x + b$ has one fixed point and every other point lies in a 2-cycle, so there are infinitely many 2-element invariant sets.

Thus, exactly two such subsets cannot occur.

True or False

1. EXPLANATION:

Consider $P(x) = 2010(x + 1)^3$.

Here $x = -1$ is the only root of $P(x)$ and $|-1| < 10$.

Note that if P was monic and z_1, z_2, z_3 were the three roots, then $z_1 z_2 z_3 = -2010$, which gives $|z_1||z_2||z_3| = 2010$ and forces at least one root to have magnitude greater than 10.

2. EXPLANATION:

$\left(\frac{f(x)}{g(x)}\right)^2 = x$ implies $\big(f(x)\big)^2 = x(g(x))^2$.

But $\big(f(x)\big)^2$ is a polynomial of even degree while $x(g(x))^2$ is odd degree polynomial, a contradiction.

So, no such f and g exists.

3. EXPLANATION:

$|z_1 - z_2|^2 + |z_1 - z_4|^2 + |z_2 - z_3|^2 + |z_3 - z_4|^2$
$= (z_1 - z_2)(\overline{z_1 - z_2}) + (z_1 - z_4)(\overline{z_1 - z_4}) + (z_2 - z_3)(\overline{z_2 - z_3}) + (z_3 - z_4)(\overline{z_3 - z_4})$
$= (z_1 - z_2)(\overline{z_1} - \overline{z_2}) + (z_1 - z_4)(\overline{z_1} - \overline{z_4}) + (z_2 - z_3)(\overline{z_2} - \overline{z_3}) + (z_3 - z_4)(\overline{z_3} - \overline{z_4})$
$= 2(|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2) - 2\,Re\,(z_1\overline{z_2} + z_3\overline{z_2} + z_3\overline{z_4} + z_1\overline{z_4})$
$= 2(1) - 2\,Re\,((z_1 + z_3)(\overline{z_2 + z_4}))$
$= 2 + 2\,Re|z_1 + z_3|^2$
$= 2 + 2\,|z_1 + z_3|^2$, which is minimum when $z_1 = -z_3$, which gives minimum value 2.

4. EXPLANATION:

Take $f(z) = e^z$, $z = x + iy$. Then $|f(z)| = |e^z| = e^x \leq 1$, when $x \leq 0$.

Let γ be the straight line joining α to β.

Then $\left|e^\alpha - e^\beta\right| = \left|\int_\gamma e^z\ dz\right| \leq \int_\gamma |e^z|\ dz \leq \int_\gamma dz = \beta - \alpha \leq |\alpha - \beta|$

TIFR MATHEMATICS: NUMBER OF QUESTIONS ASKED IN TIFR EXAM (2010 to 2025)																	
CHAPTER: 4- ABSTRACT ALGEBRA		2010	2011	2012	2013	2014	2015	2016	2017	2018	2019	2020	2021	2022	2023	2024	2025
1	Number theory	3	2	2	-	1	2	-	2	1	-	1	1	1	-	-	1
2	Groups and subgroups	1	1	2	3	3	3	-	2	3	1	2	3	1	2	1	2
3	Cyclic Groups and Permutation Groups	3	-	-	1	-	-	2	1	1	1	1	-	1	-	2	-
4	Normal subgroups, factor groups, class equation	-	-	1	1	-	-	-	-	-	1	-	1	-	1	3	-
5	Homomorphism	-	1	2	1	3	1	1	-	2	-	1	1	2	2	-	2
6	Sylow Theorems and Applications	-	-	-	-	-	1	-	-	-	-	1	-	-	-		1
7	Important structures of Rings	-	1	-	-	-	-	1	-	-	3	2	-	-	2	1	-
8	Ideals and Homomorphism	-	-	-	-	-	-	-	3	1	2	1	3	2	-	1	-
9	Rings of polynomials and irreducibility	1	-	1	1	1	-	-	-	-	-	2	-	1	-		1
10	ED, PID, UFD	-	-	1	-	-	-	-	-	-	-	-	-	1	-		-

CHAPTER 4

ABSTRACT ALGEBRA

Unit 4.1: Number theory

MCQ (Multiple Choice Questions)

[TIFR 2010]

1. The last digit of 2^{80} is
   (a) 2 (b) 4
   (c) 6 (d) 8

[TIFR 2010]

2. Which of the following statements is false?
   (a) There exists a natural number which when divided by 3 leaves remainder 1 and which when divided by 4 leaves remainder 0.
   (b) There exists a natural number which when divided by 6 leaves remainder 2 and when divided by 9 leaves remainder 1.
   (c) There exists a natural number which when divided by 7 leaves remainder 1 and when divided by 11 leaves remainder 3.
   (d) There exists a natural number which when divided by 12 leaves remainder 7 and when divided by 8 leaves remainder 3.

[TIFR 2010]

3. If n and m are positive integers and $n^9 = 19m + r$, then the possible values for r modulo 19 are
   (a) Only 0. (b) Only $0, \pm 1$.
   (c) Only ± 1. (d) None of the above

[TIFR 2011]

4. The equation $x^{22} \equiv 2 \bmod 23$ has
   (a) no solutions. (b) 23 solutions.
   (c) exactly one solution. (d) 22 solutions.

[TIFR 2014]

5. What is the last digit of 97^{2013}?
   (a) 1 (b) 3
   (c) 7 (d) 9

[TIFR 2015]

6. Let $n \in N$ be a six-digit number whose base 10 expansion is of the form $abcabc$, where a, b, c are digits between 0 and 9 and a is non-zero. Then
   (a) n is divisible by 5 (b) n is divisible by 8
   (c) n is divisible by 13 (d) n is divisible by 17.

[TIFR 2015]

7. How many finite sequences $x_1, x_2, \ldots, x_m$ are there such that each $x_i = 1$ or 2, and $\sum_{i=1}^{m} x_i = 10$?
   (a) 89 (b) 91
   (c) 92 (d) 120

[TIFR 2018]

8. What are the last 3 digits of $2^{2^{2017}}$?
   (a) 072 (b) 472
   (c) 512 (d) 912

[TIFR 2022]

9. Let m be the number of positive integers n such that $1 \leq n \leq 2022$ and such that n has an odd number of (positive integer) divisors. Then m is
   (a) 22 (b) 33
   (c) 44 (d) 55

True/False

[TIFR 2011]

1. The equation $63x + 70y + 15z = 2010$ has an integral solution.

[TIFR 2012]

2. If n is not a multiple of 23, then the remainder when n^{11} is divided by 23 is ± 1.

[TIFR 2012]

3. Given any integer $n \geq 2$, we can always find an integer m such that each of the $n-1$ consecutive integers $m+2, m+3, \ldots, m+n$ are composite.

[TIFR 2017]

4. There exists a subset A of $\mathbb{N}$ with exactly five elements such that the sum of any three elements of A is a prime number.

[TIFR 2020]

5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of elements in $\{0,1\}$ such that for all positive integers n, $\sum_{i=n}^{n+9} a_i$ is divisible by 3. Then there exists a positive integer k such that $a_{n+k} = a_n$ for all positive integers n.

[TIFR 2021]

6. The set $\{n \in \mathbb{N} \mid n$ divides $a^3 - 1$, for all integers a such that $gcd\,(a, n) = 1\}$ is infinite.

[TIFR 2025]

7. For any positive integer d, the set $\{10^a - 10^b \mid a, b$ are distinct positive integers$\}$ contains a multiple of d.

Theory questions

[TIFR 2017]

1. Prove or disprove: there exists $A \subset N$ with exactly five elements, such that sum of any three elements of A is a prime number.

Answer Key

Multiple Choice Questions (MCQs)								
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
c	b	b	a	c	c	a	a	c

True or False						
1	**2**	**3**	**4**	**5**	**6**	**7**
True	True	True	False	True	False	True

EXPLANATION

Multiple Choice Questions (MCQs)

1. EXPLANATION:

$2^5 \equiv 2(\text{mod}10)$
$\therefore (2^5)^5 \equiv 2^5(\text{mod}10)$
$2^{25} \equiv 2(\text{mod}10)$
$2^{75} \equiv 8(\text{mod}10)$
$2^{75}2^5 \equiv 8 \cdot 2^5(\text{mod}10)$
$2^{80} \equiv 6(\text{mod } 10)$

2. EXPLANATION:

For (b), note that
As 3 divides 6, the same number when divided by 3 will leave remainder 2.
Also 3 divides 9, so the same number when divided by 3 will leave remainder 1, a contradiction.

3. EXPLANATION:

$n^9 = 19m + r$
$r = n^9 - 19m$
r modulo $19 = n^9(\text{mod}19)$
$n^{19} = n(\text{mod}19)$ (by Fermat's little theorem)
$\therefore 19|n^{19} - n$ or $19|n(n^{18} - 1)$
$19|n(n^9 - 1)(n^9 + 1)$
i.e.
$n \equiv 0\ (\text{mod}19)$
$n^9 \equiv 1(\text{mod}19)$
$n^9 = -1(\text{mod}19)$

4. EXPLANATION:

If x is multiple of 23, clearly no solution.
If x is not a multiple of 23, then for prime p, $p \mid x^p - x$ (by Fermat's little theorem)
$\Rightarrow x^p - x \equiv 0(\text{mod}\, p)$
$\Rightarrow x^{23} - x \equiv 0(\text{mod}\, p)$
$\Rightarrow x(x^{22} - 1) \equiv 0(\text{mod}\ 23)$
$\Rightarrow 23 \mid x^{22} - 1$ or $23 \mid x$
So, $x^{22} \equiv 2(\text{mod}23)$ has no solutions.

5. EXPLANATION:

$97 \equiv 7(\text{mod}10)$
$97^2 \equiv 49(\text{mod}10)$
$97^2 \equiv -1(\text{mod}10).$
$97^{2\times1006} \equiv (-1)^{1006}(\text{mod}10)$
$97^{2012} \equiv 1(\text{mod}10)$
$97 \cdot 97^{2012} \equiv 97(\text{mod}10)$
$97^{2013} \equiv 7(\text{mod}10)$

6. EXPLANATION:

$abcabc$
$= 100000a + 10000b + 1000c + 100a + 10b + c$
$= 100100a + 10010b + 1001c$
$= 1001(100a + 10b + c)$
As, 13 divides 1001, 13 divides n.

7. EXPLANATION:

Let g_n be the number of sequences with sum n made up of 1's and/or 2's.
Note that
$g_{n+1} = g_n + g_{n-1}$
For a sequence with sum $n+1$ is obtained by either appending a 1 to a sequence with sum n, or by appending a 2 to a sequence of length $n-1$.
We need to find g_{10}.
We have $g_1 = 1, g_2 = 2, g_3 = 3 \ldots$ and so on we get $g_{10} = 89$.

8. EXPLANATION:

Using $ab\ mod\ ac = a(b\ mod\ c)$,we have
$2^{2017}\ mod\ 1000 = 8\left(\frac{2^{2017}}{8}\ mod\ 125\right)$
Now we will find 2^{2017} mod 125.
$\phi(125) = 100.$
So, by Euler's theorem, we have
$2^{100} \equiv 1 \text{ mod } 125$
$\therefore 2^{2000} \equiv 1^{20}\ mod\ 125$
$\therefore 2^{2017} \equiv 2^{17}\ mod\ 125$
$\therefore 2^{2017} \equiv 2^7 \cdot 2^7 \cdot 2^3\ mod\ 125$
$\therefore 2^{2017} \equiv 3 \cdot 3 \cdot 8\ mod\ 125$
$\therefore 2^{2017} \equiv 72\ mod\ 125$
$2^{2017}\ mod\ 1000 = 8\left(\frac{2^{2017}}{8}\ mod\ 125\right) = 8\left(\frac{72}{8}\right) = 72$

9. EXPLANATION:

$1 \leqslant n \leqslant 2022$
$\Rightarrow n = p_1^{k_1}p_2^{k_2} \ldots p_r^{kr}$
$\Rightarrow \tau(n) = (k_1 + 1)(k_2 + 1)\cdots(k_r + 1)$
As, n has an odd number of divisors, each k_i is even, implies n is perfect square.
Note that $44^2 = 1936$ and $45^2 = 2025$, $m = 44$.

True/False

1. EXPLANATION:

Since $(63, 70, 15) = 1$, there exists $m, n, k \in \mathbb{Z}$, such that $63m + 70n + 15k = 1$.
Multiplying throughout by 2010, gives required x, y, z.

2. EXPLANATION:

By Fermat's theorem,
$n^{22} \equiv 1\ (\text{mod}23)$
$(n^{11})^2 - 1 \equiv 0\ (\text{mod}23)$
$(n^{11} - 1)(n^{11} + 1) \equiv 0\ (\text{mod}23)$
So, 23 either divides $n^{11} - 1$ or $n^{11} + 1$. Hence, the result.

CHAPTER 6

METRIC SPACES & TOPLOGY

Unit 6.1: Metric spaces and properties

MCQ (Multiple Choice Questions)

[TIFR 2014]

1. X is a metric space. Y is a closed subset of X such that the distance between any two points in Y is at most 1. Then
   (a) Y is compact
   (b) any continuous function from $Y \to R$ is bounded
   (c) Y is not an open subset of X
   (d) none of the above.

[TIFR 2015]

2. Let $d(x,y)$ be the usual Euclidean metric on $\mathbb{R}^2$. Which of the following metric spaces is complete?
   (a) $\mathbb{Q}^2 \subset \mathbb{R}^2$ with the metric $d(x,y)$
   (b) $[0,1] \times [0,\infty) \subset \mathbb{R}^2$ with the metric $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$
   (c) $(0,\infty) \times [0,\infty) \subset \mathbb{R}^2$ with the metric $d(x,y)$
   (d) $[0,1] \times [0,1) \subset \mathbb{R}^2$ with the metric $d"(x,y) = min\{1, d(x,y)\}$.

[TIFR 2016]

3. Let (X,d) be a metric space. Which of the following is possible?
   (a) X has exactly 3 dense subsets
   (b) X has exactly 4 dense subsets
   (c) X has exactly 5 dense subsets
   (d) X has exactly 6 dense subsets.

[TIFR 2016]

4. For $X \subset \mathbb{R}^n$, consider X as a metric space with metric induced by the usual Euclidean metric on $\mathbb{R}^n$. Which of the following metric spaces X is complete?
   (a) $X = \mathbb{Z} \times \mathbb{Z} \subset \mathbb{R} \times \mathbb{R}$
   (b) $X = \mathbb{Q} \times \mathbb{R} \subset \mathbb{R} \times \mathbb{R}$
   (c) $X = (-\pi, \pi) \cap \mathbb{Q} \subset \mathbb{R}$
   (d) $X = [-\pi, \pi] \cap (\mathbb{R} \setminus Q) \subset \mathbb{R}$

[TIFR 2022]

5. Consider the following properties of a metric space (X,d):
   (I) (X,d) is complete as a metric space.
   (II) For any sequence $\{Z_n\}_{n\in N}$ of closed nonempty subsets of X such that $Z_1 \supseteq Z_2 \supseteq \cdots$ and $\lim_{n\to\infty}(\sup_{x,y\in Z_n} d(x,y)) = 0$, $\bigcap_{n=1}^{\infty} Z_n$ is a singleton set.
   Which of the following sentences is true?
   (a) (I) implies (II) and (II) implies (I).
   (b) (I) implies (II) but (II) does not imply (I).
   (c) (I) does not imply (II) but (II) implies (I).
   (d) (I) does not imply (II) and (II) does not imply (I).

[TIFR 2023]

6. Consider pairs (X,d), where X is a set with 100 elements, and $d: X \times X \to R$ is a function such that $d(x,y) = d(y,x) > 0$ if $x, y \in X$ are distinct, and $d(x,x) = 0$ for all $x \in X$. For $n < 100$, let A_n be the statement:
   For every such pair (X,d), there exists a subset X_1 of X, with n elements, such that $(X_1, d_{X_1 \times X_1})$ is a metric space.
   Which of the following statements is correct?
   (a) A_2 is true, but A_3 is not true.
   (b) A_3 is true, but A_4 is not true.
   (c) A_n is true for all $n \leq 10$, but not for all $n \leq 25$.
   (d) A_n is true for all $n \leq 25$.

True/False

[TIFR 2013]

1. There exists a complete metric on the open interval $(0,1)$ inducing the usual topology).

[TIFR 2017]

2. For any metric space (X,d) with X finite, there exists an isometric embedding $f: X \to \mathbb{R}^4$.

[TIFR 2019]

3. If 0 is a limit point of a set $A \subseteq (0,\infty)$ then the set of all $x \in (0,\infty)$ that can be expressed as a sum of (not necessarily distinct) elements of A is dense in $(0,\infty)$.

[TIFR 2020]

4. Let $C([0,1])$ denote the metric space of continuous real valued functions on $[0,1]$ under the supremum metric - i.e., the distance between f and g in $C([0,1])$ equals $\sup\{|f(x) - g(x)| x \in [0,1]\}$. Let $Q \subset C([0,1])$ be the set of all polynomials in $\mathbb{R}[x]$ in which the coefficient of x^2 is 0. Then Q is dense in $C([0,1])$.

[TIFR 2020]

5. Let X be a metric space, and let $C(X)$ denote the $\mathbb{R}$-vector space of continuous real valued functions on X. Then X is infinite if and only if $\dim_{\mathbb{R}} C(X) = \infty$.

[TIFR 2021]

6. Every infinite closed subset of R^n is the closure of a countable set.

[TIFR 2021]

7. Define a metric on the set of finite subsets of Z as follows: $d(A,B) =$ cardinality of $(A \cup B \setminus (A \cap B))$. The resulting metric space admits an isometry into R^n, for some positive integer n.

[TIFR 2021]

8. Let (X, d) be an infinite compact metric space. Then there exists no function $f : X \to X$, continuous or otherwise, with the property that $d\big(f(x), f(y)\big) > d(x, y)$ for all $x \neq y$.

[TIFR 2021]

9. If X is a compact metric space, there exists a surjective (not necessarily continuous) function $\mathbb{R} \to X$.

[TIFR 2021]

10. If X is a compact metric space, then every isometry $f : X \to X$ is surjective.

[TIFR 2022]

11. There exists a metric space (X, d) such that the group of isometries of X is isomorphic to $\mathbb{Z}$.

[TIFR 2023]

12. If A and B are disjoint subsets of a metric space (X, d), then $\inf\{x \in A, y \in B\} \neq 0$.

[TIFR 2023]

13. A countably infinite complete metric space has infinitely many isolated points (an element x of a metric space X is said to be an isolated point if $\{x\}$ is an open subset of X).

[TIFR 2024]

14. There exists a metric space X such that the number of open subsets of X is exactly 2024.

[TIFR 2024]

15. The function $d: \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ given by $d(x, y) = |e^x - e^y|$ defines a metric on $\mathbb{R}$, and $(\mathbb{R}, d)$ is a complete metric space.

[TIFR 2024]

16. $\mathbb{Q} \cap [0, 1]$ is a compact subset of $\mathbb{Q}$.

[TIFR 2025]

17. Let (X, d) be a nonempty complete metric space, and let $T: X \to X$ be a continuous map such that T^2 is a contraction, i.e., there exists $a < 1$ such that for all distinct $x, y \in X$, we have $d(T^2(x), T^2(y)) < \mathrm{ad}(x, y)$. Then there exists a unique element $x \in X$ such that $T(x) = x$.

[TIFR 2025]

18. Let $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$, and let $d: S^2 \times S^2 \to \mathbb{R}$ be the restriction of the Euclidean metric on $\mathbb{R}^3$. If $f: S^2 \to S^2$ is a map such that $d(f(x), f(y)) = d(x, y)$ for all $x, y \in S^2$, then f is surjective.

Theory questions

[TIFR 2017]

1. Show that the subset $GL_n(\mathbb{R})$ of $M_n(\mathbb{R})$ consisting of all invertible matrices is dense in $M_n(\mathbb{R})$.

Answer Key

Multiple Choice Questions (MCQs)					
1	**2**	**3**	**4**	**5**	**6**
d	b	b	a	a	a

True or False									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
True	False	True	True	True	True	False	True	True	True
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**		
True	False	True	False	False	False	True	True		

EXPLANATION

Multiple Choice Questions (MCQs)

1. EXPLANATION:

Let $X = \mathbb{R}$.

Define $d: X \times X \to \mathbb{R}$, a discrete metric such that

$d(x,y) = \begin{cases} 0 & x = y \\ 1 & x \neq y \end{cases}$

Let $Y = \mathbb{Q}$.

Since every subset of discrete metric space is open, option (c) is incorrect.

Also $\bigcup_{r\in\mathbb{Q}}\{r\}$ is an open cover of Y, which does not admit a finite subcover because Y is infinite.

So, option (a) is incorrect.

Consider $f: Y \to \mathbb{R}$, $\mathbb{R}$ with usual metric, such that $f(x) = x$.

Then f is continuous because under discrete metric every function is continuous but f is not bounded.

This discards option (b).

So, option (d) is correct.

2. EXPLANATION:

Note that metrics d, d', d" are equivalent, so we can check options (b), (d) with respect to metric d itself.

We know that $\mathbb{R}^2$ with usual metric is complete.

Also note that a subset of a complete metric space is complete if and only if it is closed.

So, we just need to check which of the given subsets of $\mathbb{R}^2$ are closed with respect to metric d.

Subsets in options (a), (c), (d) are not closed as

$\overline{\mathbb{Q}^2} = \mathbb{R}^2$, $\overline{(0,\infty)\times[0,\infty)} = [0,\infty) \times [0,\infty)$, $\overline{[0,1]\times[0,1)} = [0,1]\times[0,1]$.

Therefore, option (b) is correct.

3. EXPLANATION:

If a space X contains only one of its limit points, say a, then it will have two dense subsets, which are X and $X\backslash\{a\}$. For e.g., if $X = \left\{\frac{1}{n} : n \in \mathbb{N}\right\} \cup \{0\}$, its dense subsets are X and $X\backslash\{0\}$.

If a space X contains two of its limit points, say a and b, then it will have four dense subsets, which are X, $X\backslash\{a\}$, $X\backslash\{b\}$ and $X\backslash\{a,b\}$.

Generalizing this, we have, if a space X contains n of its limit points, then it will have 2^n dense subsets.

Thus, option (b) is correct.

4. EXPLANATION:

We know that $\mathbb{R}^n$ with usual metric is complete.

Also note that a subset of a complete metric space is complete if and only if it is closed.

So, we just need to check which of the given subsets of $\mathbb{R}^n$ are closed with respect to metric d.

Subsets in options (b), (c), (d) are not closed as

$\overline{\mathbb{Q}\times\mathbb{R}} = \overline{\mathbb{R}\times\mathbb{R}}$, $\overline{(-\pi,\pi)\cap\mathbb{Q}} = [-\pi,\pi]$, $\overline{[-\pi,\pi]\cap(\mathbb{R}\setminus Q)} = [-\pi,\pi]$.

Thus, option (a) is correct.

5. EXPLANATION:

(I) ⇒ (II)

Note that $\sup_{x,y\in\mathbb{Z}_n} d(x,y) = diam\ Z_n$

Each Z_n is closed (given) and there must exist n such that for all $m \geq n$, diam(Z_m) is finite because of given limit condition.

X being complete, Z_m's $(m \geq n)$ are compact as they are closed and bounded.

Then using finite intersection property, $\bigcap_{n=1}^{\infty} Z_n$ is non-empty and it has to be singleton otherwise $\lim_{n\to\infty}(\sup_{x,y\in\mathbb{Z}_n} d(x,y))$ cannot be zero.

(II) ⇒ (I)

For Cauchy sequence x_n in X, consider $Z_n = \{x_m : m \geq n\}$. Then sequence $\{\overline{Z_n}\}$ satisfies the hypotheses of (II) and x_n converges to the intersecting point $\bigcap_{n=1}^{\infty} Z_n$.

So, every Cauchy sequence is convergent and X is complete.

6. EXPLANATION:

If $|X_1| = 2$, say $X_1 = \{a,b\}$, then $\left(X_1, d_{X_1\times X_1}\right)$ is a metric space as by given conditions

(i) $d(a,b) > 0$

(ii) $d(a,b) = d(b,a)$

(iii) $d(a,a) = d(b,b) = 0$

and

(iv) Triangle inequality is also true as $|X_1| = 2$

Therefore A_2 is true.

But A_3 is not necessarily true. The idea is as follows.

Consider $d(x,y) = [|x-y|]$.

Then clearly $d(x,y) = d(y,x) > 0$ when $x \neq y$ and $d(x,x) = 0\ \forall x$

Consider $X_1 = \{1, \frac{5}{2}, 4\}$ with above d

Here $d(1,4) = 3$ but

$3 = d(1,4) > d\left(1,\frac{5}{2}\right) + d\left(\frac{5}{2},4\right) = 1+1 = 2$

contrary to triangle inequality.