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SECTION – I

INTEGRAL CALCULUS

CHAPTER 1

INDEFINITE INTEGRATION

1.1 Definition: - Integration is inverse process of differentiation the process of finding $f(x)$ when it's derivative $f'(x)$ is given is known as integration.

Integration as anti-derivative: -
If $f(x)$ is differentiable function such that $f'(x) = g(x)$ then integration $f\ g(x)$ w.r.t x is $f(x) + c$
Mathematically,
$\int g(x)dx = f(x) + c$

Example $\frac{d}{dx}(\tan x) = \sec^2 x$
$\int \sec^2 x\, dx = \tan x + c$ integral constant

List of Standard Formula

$f(x)$	$\int f(x)\, dx$
1. k	$kx + c$
2. $x^n\ (n \neq -1)$	$\frac{x^{n+1}}{n+1} + c$
3. $\frac{1}{x}\ (x \neq 0)$	$\log\|x\| + c$
4. $a^x\ (a > 0)$	$\frac{a^x}{\log a} + c$
5. e^x	$e^x + c$
6. $\sin x$	$-\cos x + c$
7. $\cos x$	$\sin x + c$
8. $\sec^2 x$	$\tan x + c$
9. $cosec^2\ x$	$-\cot x + c$
10. $\sec x \cdot \tan x$	$\sec x + c$
11. $cosec\ x \cdot \cot x$	$-cosec\ x + c$
12. $\frac{1}{\sqrt{1-x^2}}$	$\sin^{-1} x + c$
13. $\frac{1}{1+x^2}$	$\tan^{-1} x + c$
14. $\frac{1}{x\sqrt{x^2-1}}$	$\sec^{-1} x + c$

1.2 Properties of Indefinite Integration:
(1) $\int a\, f(x) \pm b\, g(x)\, dx = a\int f(x)dx \pm b\int g(x)\, dx$
(2) If $\int f(x)\, dx = g(x) + c$ then $\int f(ax + b)dx = \frac{g(ax+b)}{a} + c$

Example 1: $\int \frac{(\sqrt{x}+1)(x^2-\sqrt{x})}{x\sqrt{x}+x+\sqrt{x}}dx.$

Solution: Here,

$$I = \int \frac{(\sqrt{x}+1).\sqrt{x}(x^{3/2}-1)}{\sqrt{x}(x+\sqrt{x}+1)}dx$$

$$\therefore I = \int \frac{(\sqrt{x}+1)\left[(\sqrt{x})^3-1^3\right]}{(x+\sqrt{x}+1)}dx$$

$$= \int \frac{(\sqrt{x}+1)(\sqrt{x}-1)(x+\sqrt{x}+1)}{(x+\sqrt{x}+1)}dx \qquad [\text{Since, } a^3-b^3=(a-b)(a^2+ab+b^2)]$$

$$= \int (x-1)\,dx = \frac{x^2}{2} - x + C$$

Example 2: $\int \sec^2(8x+5)\,dx$

Solution: $\int \sec^2 x\,dx = \tan x + c$

$$\therefore \int \sec^2(8x+5)\,dx = \frac{\tan(8x+5)}{8} + c$$

1.3 Methods of Integration:

If the integrand is not a derivative of a known function, then the corresponding integral cannot be found directly.

In order to find the integral of complex problems, generally five rules of integration are used.

(a) Integration by substitution.

(b) Integration by parts.

(c) Integration by partial fractions.

(d) Integration Using Differentiation

(e) special integral's

1.3.1 Integration by substitution:

(i) Direct Substitution:

Remark:

(1) $\int \frac{f'(x)}{f(x)}dx = In|f(x)| + c$

(2) $\int [f(x)]^n f'(x)dx = \frac{[f(x)]^{n+1}}{n+1} + c, (n \neq -1)$

Example:

$I = \int \cot x\,dx$

$\Rightarrow I = \int \frac{\cos x}{\sin x}dx$

Put $\sin x = t$

$\cos x = \frac{dt}{dx} \qquad \Rightarrow \cos x\,dx = dt$

$I = \int \frac{dt}{t}$

$= \log|t| + c$

$= \log|\sin x| + c$

$\int \cot x\,dx = \log|\sin x| + c$

Similarly, $\int \tan x\,dx = \log|\sec x| + c$

Example:

Evaluate: $I = \int \sqrt{\frac{1-x}{1+x}}dx$

(a) $\sin^{-1} x - \sqrt{1-x^2} + c$ **(b)** $\sin^{-1} x + \sqrt{\mathbf{1+x^2}} + \boldsymbol{c}$

(c) $\sin^{-1} x + \sqrt{\mathbf{1-x^2}} + \boldsymbol{c}$ **(d)** $\sin^2 x - \sqrt{1+x^2} + c$

Solution:

$I = \int \sqrt{\frac{1-x}{1+x}}\,dx$

$= \int \frac{\sqrt{1-x}}{\sqrt{1+x}} \cdot \frac{\sqrt{1-x}}{\sqrt{1-x}}\,dx$

$= \int \frac{1-x}{\sqrt{1-x^2}}\,dx$

$= \int \frac{dx}{\sqrt{1-x^2}}\,dx$

$= \int \frac{dx}{\sqrt{1-x^2}} - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$= \sin^{-1} x + c_1 - \int \frac{x}{\sqrt{1-x^2}}\,dx$

Now, $= \int \frac{x}{\sqrt{1-x^2}}\,dx$

Let $1 - x^2 = z$

$-2xdx = dz$

$xdx = -\frac{dz}{2}$

$\therefore \int \frac{x}{\sqrt{1-x^2}}\,dx = -\frac{1}{2}\int \frac{dz}{\sqrt{z}}$

$= -\frac{1}{2} \times 2\sqrt{z}$

$= -\sqrt{z} + c_2$

$\therefore I = \sin^2 x + \sqrt{1-x^2} + c$ (Ans.)

Example :

Evaluate $\int \sec^{\frac{2}{3}} x \ cosec^{\frac{4}{3}} x\,dx$

Solution:

$\int \sec^{\frac{2}{3}} x \ cosec^{\frac{4}{3}} x\,dx \ = \int \frac{\sec^{\frac{2}{3}} x}{\sin^{\frac{4}{3}} x}\,dx = \int \frac{\sec^2 x}{\tan^{\frac{4}{3}} x}\,dx$

Put $\tan x = t \quad \Rightarrow \quad \sec^2 x \ dx = dt$

$\Rightarrow -3\tan^{-\frac{1}{3}} x + c$

(ii) Standard Substitution:

Integrand	**Substitution**
1) $x^2 + a^2$ or $\sqrt{x^2 + a^2}$	$x = a\tan\theta$
2) $x^2 - a^2$ or $\sqrt{x^2 - a^2}$	$x = a\sec\theta$
3) $a^2 - x^2$ or $\sqrt{a^2 - x^2}$	$x = a\sin\theta$ or $a\cos\theta$
4) $\sqrt{a+x}$ & $\sqrt{a-x}$	$x = a\cos 2\theta$
5) $\left(x \pm \sqrt{x^2 \pm a^2}\right)^n$	the expression within the bracket = t
6) $\frac{2x}{a^2-x^2}, \frac{2x}{a^2+x^2}, \frac{a^2-x^2}{a^2+x^2}$	$x = a\tan\theta$
7) $\frac{1}{(x+a)^{1-\frac{1}{n}}(x+b)^{1+\frac{1}{n}}} (n > 1)$	$\frac{x+a}{x+b} = t$
8) $\sqrt{2ax - x^2}$	$x = a(1 - \cos\theta)$
9) $\frac{1}{\sqrt{(x-a)(x-b)}}$	$x - a = t^2$ or $x - b = t^2$
10) $\frac{1}{(x+a)^{n_1}(x+b)^{n_2}}\ n_1, n_2 \in \mathbb{N}$(and > 1)	$(x + a) = t(x + b)$

Algebraic Integral's:

(i) $\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right) + c$

(ii) $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right| + c$

(iii) $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right) + c$

(iv) $\int \frac{dx}{\sqrt{a^2+x^2}} = \ln\left(x + \sqrt{a^2+x^2}\right) + c$

(v) $\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x + \sqrt{x^2-a^2}\right) + c$

(vi) $\int \sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\left(\frac{x}{a}\right) + c$

(vii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\left(\sqrt{x^2+a^2}\right) + \frac{a^2}{a}\ln\left(x + \sqrt{x^2+a^2}\right) + c$

(viii) $\int \sqrt{x^2-a^2}\,dx = \frac{x}{2}\sqrt{x^2-a^2} - \frac{a^2}{2}\ln\left(x + \sqrt{x^2-a^2}\right) + c$

Example 1: $\int \frac{dx}{(x+3)^{\frac{15}{16}}(x-4)^{\frac{17}{16}}}$

(a) $\frac{16}{7}\left\{\frac{x+3}{x-4}\right\}^{\frac{1}{16}} + c$ (b) $\frac{-16}{7}\left\{\frac{x+3}{x-4}\right\}^{\frac{1}{16}} + c$

(c) $\frac{7}{16}\left\{\frac{x+3}{x-4}\right\}^{\frac{1}{16}} + c$ (d) $\frac{1}{7}\left\{\frac{x+3}{x-4}\right\}^{\frac{1}{16}} + c$

Solution:

Here, $\int \frac{dx}{(x+3)^{\frac{15}{16}}(x-4)^{\frac{17}{16}}}$

Put $\frac{x+3}{x-4} = t$

We know that $\left(\frac{u}{v}\right)' = \frac{vu'-uv'}{v^2}$

$\frac{(x-4)1-(x+3)1}{(x-4)^2}\,dx = dt$

$\frac{1}{(x-4)^2}dx = \frac{dt}{-7}$

$\int \frac{(x-4)^2 dx}{(x+3)^{1-\frac{1}{16}}(x-4)^{1+\frac{1}{16}}}$

$= \int \frac{(x-4)^2}{(x+3)^{\frac{15}{16}}(x-4)^{\frac{17}{16}}(x-4)^2}\,\frac{dt}{-7}$ (adjusting $(x-4)^2$)

$= \frac{-1}{7}\int \frac{dt}{(x+3)^{\frac{15}{16}}(x-4)^{-\frac{15}{16}}}$

$= -\frac{1}{7}\int \frac{dt}{\frac{(x+3)^{\frac{15}{16}}}{(x-4)^{\frac{15}{16}}}} = -\frac{1}{7}\int \frac{dt}{\left(\frac{x+3}{x-4}\right)^{\frac{15}{16}}}$

$= -\frac{1}{7}\int \frac{dt}{(t)^{\frac{15}{16}}}$

$= -\frac{1}{7}\int t^{-\frac{15}{16}}dt$

$= -\frac{1}{7}\left\{\frac{t^{-\frac{15}{16}+1}}{-\frac{15}{16}+1}\right\} + c$

$= -\frac{1}{7}\left\{\frac{t^{\frac{1}{16}}}{\frac{1}{16}}\right\} + c$

$= -\frac{16}{7}\left\{\frac{x+3}{x-4}\right\}^{\frac{1}{16}} + c$

Example 2: $\int \frac{dx}{\left(X+\sqrt{X^2-3}\right)^{7/2}}$

Solution:

$\int \frac{dX}{\left(X+\sqrt{X^2-3}\right)^{7/2}}$

Put $X+\sqrt{X^2-3}=t$

$\Rightarrow \sqrt{X^2-3}=t-X$

$\Rightarrow X^2-3=t^2-2tX+X^2$

$\Rightarrow t^2=2tX-3$

$\Rightarrow t^2=2tX-3$

$\Rightarrow 2tX=t^2-3$

$\Rightarrow X=\frac{t}{2}-\frac{3}{2t}$

$\Rightarrow dX=\left(\frac{1}{2}+\frac{3}{2t^2}\right)dt$

$\therefore I=\int \frac{\left(\frac{1}{2}+\frac{3}{2t^2}\right)}{t^{\frac{7}{2}}}dt$

$=\frac{1}{2}\int t^{\frac{7}{3}}dt+\frac{3}{2}\int t^{-\frac{11}{2}}dt$

$=\frac{1}{2}\frac{t^{\frac{10}{3}}}{\frac{10}{3}}+\frac{3}{2}\frac{t^{-\frac{9}{2}}}{-\frac{9}{2}}+C$

$=\frac{3}{20}t^{\frac{10}{3}}-\frac{1}{3}t^{-\frac{9}{2}}+c$

$=\frac{3}{20}\left(X+\sqrt{X^2-3}\right)^{\frac{10}{3}}-\frac{1}{3}\left(X+\sqrt{X^2-3}\right)^{-\frac{9}{2}}+C$

(iii) Indirect Substitution:

If integrand $f(x)$ can be rewritten as product of two function i.e. $f(x)=f_1(x).f_2(x)$ where $f_2(x)$ is a function of integral of $f(x)$

Example: $\int \sqrt{\frac{x}{4-x^3}}dx$

$\sqrt{\frac{x}{4-x^3}}=\sqrt{x}\times\frac{1}{\sqrt{4-x^3}}$

$f_1=\sqrt{x}$

$f_2=\frac{1}{\sqrt{4-\left(x^{\frac{3}{2}}\right)^2}}$

Since $\int \sqrt{x}\,dx=\frac{x^{\frac{3}{2}}}{\frac{3}{2}}+c$

$put\ x^{\frac{3}{2}}=t$

$\sqrt{x}dx=\frac{2}{3}dt$

$I=\int \sqrt{\frac{x}{4-x^3}}dx$

$=\int \sqrt{x}\cdot\frac{1}{\sqrt{4-\left(x^{\frac{3}{2}}\right)^2}}dx$

$=\frac{2}{3}\int\frac{1}{\sqrt{4-t^2}}dt$

$=\frac{2}{3}\int\frac{1}{\sqrt{2^2-t^2}}dt$

$=\frac{2}{3}\left\{\sin^{-1}\left(\frac{t}{2}\right)\right\}+c$

$=\frac{2}{3}\left\{\sin^{-1}\left(\frac{x^{\frac{3}{2}}}{2}\right)\right\}+c$

1.3.2 Integration by part's:

If integral is expressed as product of functions u & v then

$\int u \cdot v\, dx = u \int v\, dx - \int \{\int v\, dx \cdot u'\}\, dx$

Remark's:

(1) Put constant C for whole integration in the end.

(2) order of u & v is decided by "ILATE" rule

I → inverse

L → Logarithmic

A → Algebraic

T → Trigonometric

E → Expenential

(3) Bernoulli's formula for Integration by Parts

If u and v are functions of x, then the Bernoulli's rule is ∫uv dv = uv − u 'v$_1$ + u ''v$_2$ - .....

where u ', u '', u''', ..are successive derivatives of u

and v, v$_1$, v$_2$, v$_3$,... are successive integrals of v

Bernoulli's formula is advantageously applied when u = x^n (n is a positive integer)

Problem: $\int x^2 \sin x\, dx$

Solution:

By ILATE Rule,

$x^2 \rightarrow Algebric \rightarrow u$

$\sin x \rightarrow Trigonometric \rightarrow v$

$\int x^2 \sin x\, dx = x^2 \int \sin x\, dx - \int \{\int \sin x\, dx\ (2x)\}\, dx$

$= x^2(-\cos x) - 2\int -\cos x\, x$

$= -x^2 \cos x + 2\int x \cos x\, dx$

$= -x^2 \cos x + 2\{x \int \cos x\, dx - \int \{\int \cos x \cdot dx\}\}$

$= -x^2 \cos x + 2\{x \sin x + \cos x\}$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$

Or

By Bernoulli's formula

$\int x^2 \sin x\, dx$

∫uv dx = uv − u 'v$_1$ + u ''v$_2$ - .....

$= x^2(-\cos x) - (2x)(-\sin x) + 2(\cos x) + c$

$= -x^2 \cos x + 2x \sin x + 2 \cos x + c$

Example: $\int x \cos x\, dx$

$= x(\sin x) - 1(-\cos x)$

$= x \sin x + \cos x + c$

Example: Evaluate: $\int \frac{x^2 \sin^{-1} x}{(1-x^2)^{5/2}} dx$

(a) $\frac{1}{3}\left[\left(\frac{x}{\sqrt{1-x^2}}\right)^3 \sin^{-1} x + \frac{1}{2}\frac{x^2}{1-x^2} - \frac{1}{2}\log|1-x^2|\right] + c$

(b) $\frac{1}{3}\left[\left(\frac{x}{\sqrt{1-x^2}}\right)^3 \sin^{-1} x - \frac{1}{2}\frac{x^2}{1-x^2} + \frac{1}{2}\log|1-x^2|\right] + c$