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SECTION-I

NBHM M.Sc

REAL ANALYSIS

CHAPTER 1

UNIT 1.1: FUNCTIONS AND THEIR PROPERTIES

(MSc NBHM-2011)

1. Given that the sum of two of its roots is zero, solve the equation:

$$6x^4 - 3x^3 + 8x^2 - x + 2 = 0$$

(MSc NBHM-2012)

2. Pick out the true statements:
   a. $|\sin x - \sin y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   b. $|\sin 2x - \sin 2y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.
   c. $|\sin^2 x - \sin^2 y| \leq |x - y|$ for all $x, y \in \mathbb{R}$.

(MSc NBHM-2012)

3. Let $x > 0$. Fill in the blanks with the correct sign $>, \geq, <$ or $\leq$:
   a. $\tan^{-1} x \ldots . \frac{x}{1+x^2}$
   b. $\log(1 + x) \ldots \frac{x}{1+x}$

(MSc NBHM-2014)

4. Given a function $u: \mathbb{R} \to \mathbb{R}$, define $u^+(x) = \max\{u(x), 0\}$ and $u^-(x) = -\min\{u(x), 0\}$. If u_1 and u_2 are real valued functions defined on $\mathbb{R}$, which of the following statements are true?
   (a) $|u_1 - u_2| = (u_1 - u_2)^+ + (u_1 - u_2)^-$
   (b) $\max\{u_1, u_2\} = (u_1 - u_2)^+ + u_2$
   (c) $\max\{u_1, u_2\} = (u_1 - u_2)^- + u_1$

(MSc NBHM-2014)

5. Let $x, y \in]0, \infty[$. Which of the following statements are true?
   (a) $|\log(1 + x^2) - \log(1 + y^2)| \leq |x - y|$.
   (b) $|\sin^2 x - \sin^2 y| \leq |x - y|$.
   (c) $|\tan^{-1} x - \tan^{-1} y| \leq |x - y|$

(MSc NBHM-2016)

6. Which of the following statements are true?
   a. $|\sin x| \leq |x|$ for all $x \in \mathbb{R}$.
   b. $|1 - \cos x| \leq |x|$ for all $x \in \mathbb{R}$.
   c. $|\tan^{-1} x| \leq |x|$ for all $x \in \mathbb{R}$.

(MSc NBHM-2017)

7. Solve the equation: $x^4 - 10x^3 + 26x^2 - 10x + 1 = 0$.

(MSc NBHM-2017)

8. Let $n \in \mathbb{N}$. Define

$$\Lambda(n) = \begin{cases} \log p \ ; n = p^r, p \text{ a prime and } r \in \mathbb{N} \\ 0 \ ; \text{ otherwise} \end{cases}$$

Given $N \in \mathbb{N}$, compute $\sum_{d|N} \Lambda(d)$, where the sum ranges over all divisors of N.

(MSc NBHM-2018)

9. The polynomial $x^3 - 11x^2 + ax - 36$ has a positive root which is the product of the other two roots. Find the value of a.

(MSc NBHM-2018)

10. Fill in the blank below with the correct symbol $(<, \leq, >$ or $\geq)$: $tan^{-1}x \ldots \frac{x}{1+x^2}$, for all $x > 0$

(MSc NBHM-2018)

11. Let X be a non-empty set and let $A \subset X$. Define

$$X_A(x) = \begin{cases} 1, \text{if } x \in A, \\ 0, \text{if } x \notin A \end{cases}$$

If E and F are subsets of X, find a subset G of X such that $|X_E - X_F| = X_G$

(MSc NBHM-2019)

12. A degree 3 polynomial $f(x)$ with real coefficients satisfies $f(1) = 2, f'(2) = 2, f''(2) = 2$, and $f'''(2) = 12$, where $f'(x), f''(x)$, and $f'''(x)$ are the first, second, and third derivatives of $f(x)$ respectively. What is $f(2)$?

(MSc NBHM-2019)

13. What are the maximum and minimum values in the region $\{(x, y) | x^2 + y^2 \leq 1, x + y \leq 0\}$ of the function $f(x, y) = x^2 + y$?

EXPLANATIONS

1. Explanation:

The given equation is $6x^4 - 3x^3 + 8x^2 - x + 2 = 0$

Given: $\alpha + \beta = 0$ Hence, $\alpha = -\beta$

Now, from the given equation

$$\alpha + \beta + \gamma + \delta = 3/6 = 1/2$$
$$(\alpha + \beta)(\gamma + \delta) + \alpha\beta + \gamma\delta = 8/6 = 4/3$$
$$(\alpha + \beta)\gamma\delta + \alpha\beta(\gamma + \delta) = 1/6$$
$$\alpha\beta\gamma\delta = 2/6 = 1/3$$

Applying the given condition, the above equations become,

$$\gamma + \delta = 1/2 \cdots \text{(I)}$$
$$\alpha\beta + \gamma\delta = 4/3 \cdots \text{(II)}$$
$$\alpha\beta(\gamma + \delta) = 1/6 \cdots \text{(III)}$$

$$\alpha\beta\gamma\delta = 1/3 \cdots \text{(IV)}$$

using I and III, we get, $\alpha\beta = 1/3 \cdots$ (V)

using IV and V, we get, $\gamma\delta = 1 \cdots$ (VI)

So, now we have 2 quadratic equations,

$$x^2 - 0x + 1/3 = 0(\text{ for } \alpha, \beta) \text{ and } x^2 - 1/2x + 1 = 0(\text{ for } \gamma, \delta)$$

i.e. $3x^2 + 1 = 0$ and $2x^2 - x + 2 = 0$

Solving, the quadratic equations, we get, the roots as

$$\frac{i}{\sqrt{3}}, -\frac{i}{\sqrt{3}}, \frac{1 + i\sqrt{15}}{4}, \frac{1 - i\sqrt{15}}{4}$$

2. Explanation:

(a) Consider $f(x) = \sin x$.

$\therefore |f'(x)| = |\cos x| \leq 1$

$\therefore f(x)$ is Lipschitz with $K = 1$ and hence inequality follows.

So, (a) is correct.

(b) Consider $f(x) = \sin 2x$.

$\therefore |f'(x)| = |2\cos 2x| \leq 2$

$\therefore f(x)$ is Lipschitz with $K = 2$.

So, (b) is incorrect.

(c) Consider $f(x) = \sin^2 x$.

$\therefore |f'(x)| = |\sin 2x| \leq 1$

$\therefore f(x)$ is Lipschitz with $K = 1$ and hence inequality follows.

So, (c) is correct.

3. Explanation:

(a) $f(1) = \tan^{-1}(1) = \frac{\pi}{4} > \frac{1}{2} = g(1)$ and as only one symbol is correct

$\therefore \tan^{-1} x > \frac{x}{1+x^2}$

(or you can also show $f(x)$ is increasing by showing $f'(x) > 0$ and then taking $f(x) > f(0)$ for $x > 0$)

(b) $f(1.7) = \log(2.7) \approx 1 > \frac{1.7}{1+1.7} = g(1.7)$ and as only one symbol is correct

$\therefore \log(1 + x) > \frac{x}{1+x}$

4. Explanation: All the three statements are correct using standard properties of functions. Also, it is easy to see from graphical approach.

5. Explanation:

(a) Consider f(x) $= \log(1 + x^2)$

$\therefore f'(x) = \frac{2x}{1+x^2} \leq 1$

$\therefore f'(x)$ is bounded in $(0, \infty)$

$\therefore f(x)$ is Lipschitz with $K = 1$

$\Rightarrow |\log(1 + x^2) - \log(1 + y^2)| \leq |x - y|$

So, (a) is correct.

(b) Consider $f(x) = \sin^2 x$.

$\therefore |f'(x)| = |\sin 2x| \leq 1$

$\therefore f(x)$ is Lipschitz with $K = 1$ and hence inequality follows.

So, (b) is correct.

(c) Consider $f(x) = \tan^{-1} x$.

$f'(x) = \frac{1}{1+x^2} \leq 1$

$\therefore f(x)$ is Lipschitz with $K = 1$ and hence inequality follows.

So, (c) is correct.

6. Explanation: All the statements are correct and we use same methodology for all statements.

To show $|f(x)| \leq |g(x)|$, consider $h(x) = f(x) - g(x)$.

Then show $h'(x) < 0$, i.e. h is decreasing function.

$\therefore h(x) < h(0) = 0$ for $x < 0$.

$\therefore f(x) < g(x)$ for $x < 0$.

Take modulus on both sides to get required results.

7. Explanation:

$(x^2 + ax + 1)(x^2 + bx + 1) = 0$

$x^4 + bx^3 + x^2 + ax^3 + abx^2 + ax + x^2 + bx + 1 = 0$

$x^4 + (a + b)x^3 + (2 + ab)x^2 + (a + b)x + 1$

$\Rightarrow a + b = -10$

$2 + ab = 26$

$ab = 24$

$a = \frac{24}{b}$

$a + b = -10$

$\frac{24}{b} + b = -10$

$24 + 10b + b^2 = 0$

Therefore, $a = -6 \quad b = -4$

$x^2 - 6x + 1 = 0$ or $x^2 - 4x + 1 = 0$

So, roots are $3 \pm 2\sqrt{2}$, $2 \pm \sqrt{3}$.

8. Explanation: If $N = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$, then we will consider divisors of form $p_i^{r_k}$; where $1 \leq k \leq i$.

$\sum_{d|N} \Lambda(d) = r_1 \log(p_1) + r_2 \log(p_2) + \cdots +$

$r_k \log(p_k) = \log(p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}) = \log N$

9. Explanation:

Let α, β, γ be the roots.

Say, $\alpha = \beta\gamma$ such that $\alpha > 0$.

We have $\alpha + \beta + \gamma = 11$ (ii), $\alpha\beta + \beta\gamma + \alpha\gamma = a$ (iii), $\alpha\beta\gamma = 36$ (iv)

Put $\alpha = \beta\gamma$ in eq. (iv)

$\Rightarrow \alpha^2 = 36 \Rightarrow \alpha = 6$

From eq. (ii), $6 + \beta + y = 11 \Rightarrow \beta + \gamma = 5$

From eq. (iv), $6\beta\gamma = 36 \Rightarrow \beta\gamma = 6$ … (vi)

From eq. (iii), $\alpha(\beta + \gamma)^2 + \beta\gamma = a$

Put the value of α, $\beta + \gamma$ and $\beta\gamma \Rightarrow 6(5) + 6 = a \Rightarrow a = 36$

10. Explanation:

$f(1) = \tan^{-1}(1) = \frac{\pi}{4} > \frac{1}{2} = g(1)$ and as only one symbol is correct

$\tan^{-1}x > \frac{x}{1+x^2}$

(or you can also show $f(x)$ is increasing by showing $f'(x) > 0$ and then taking $f(x) > f(0)$ for $x > 0$)

11. Explanation:

Note that if $x \in E\backslash F \cup F\backslash E$, $|(X_E - X_F)(x)| = 1$, otherwise $|(X_E - X_F)(x)| = 0$.

$\therefore G = E\backslash F \cup F\backslash E$

12. Explanation:

Consider $f(x) = a_0 + a_1x + a_2x^2 + a_3x^3$. Using given four conditions, find the four unknowns to get $f(x)$ and find that $f(2) = 5$.

13. Explanation:

$L(x, y) = x^2 + y + \lambda(x^2 + y^2 - 1) + \mu(x + y)$

$\frac{\partial L}{\partial x} = 0 \Longrightarrow 2\lambda x + \mu + 1 = 0$

$\frac{\partial L}{\partial y} = 0 \Longrightarrow 2\lambda y + \mu + 1 = 0$

$\frac{\partial L}{\partial \lambda} = 0 \Longrightarrow x^2 + y^2 += 0$

$\frac{\partial L}{\partial \mu} = 0 \Longrightarrow x\ + y\ += 0$

Solution of given system is

$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ & $\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$

$F\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right) = \frac{1}{2} + \frac{1}{\sqrt{2}}$

$F\left(\frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right) = \frac{1}{2} - \frac{1}{\sqrt{2}}$