EXAM MAPPING
NTA CSIR-UGC-DBT NET/JRF
GENERAL APTITUDE

THEORY BOOK
WITH CONCEPTS & EXPLANATIONS

NUMERICAL ABILITY | REASONING ABILITY | DATA INTERPRETATIONS | LOGICAL REASONING | PUZZLE | GRAPH

AUTHORS:

KAILASH CHOUDHARY
VARUN CHOUDHARY


SHORT TRICKS


QUICK REVISION TABLES


HIGH-YIELD DIAGRAMS


PERFECT FOR LAST 30 DAYS REVISION

IfAS | IFAS PUBLICATIONS

Copyright 2026, IFAS Publications

All rights reserved. No part of this publication or the information contained herein may be reproduced, adapted, abridged, translated, stored in a retrieval system, computer system, photographic or other systems or transmitted in any form or by any means, electronic, mechanical, optical, digital, by photocopying, recording or otherwise, without written prior permission from the publisher. Any breach will attract legal action and prosecution without further notice.

Disclaimer: While every effort has been made to avoid errors and omissions, this publication is being sold and marketed on the understanding and presumption that neither the editors (or authors) nor the publishers nor the printers would be liable in any manner whatsoever, to any person either for an error or for an omission in this publication, or for any action to be taken on the basis of this work. Any inadvertent discrepancy noted may be brought to the attention of the publisher, for rectifying it in future editions, if published.

This book contains information obtained from authentic and highly regarded sources. Reasonable efforts have been made to publish reliable data and information, but the editors and publisher cannot assume responsibility for the validity of all materials or the consequences of their use. The editors and publisher have attempted to trace and acknowledge the copyright holders of all material reproduced in this publication and apologize to copyright holders if permission and acknowledgement to publish in this form have not been obtained. If any copyright material has not been acknowledged please write and let us know so that we may rectify it.

Trademark Notice: Publications or corporate names may be trademarks, and are used only for identification and explanation in bonafide intent without intent to infringe.

Design & Published by:
IFAS Publications

Address:
Shivranjan Complex,
Someshwar Wadi, Near
Rajwada Hotel, Baner Road,
Pune 411 008.

Contact No.: **+91-960 489 7388**

Website: **https://ifasonline.com**

Price: **₹985.00**

ISBN: **978-81-69843-00-3**

Preface

Aptitude is the quiet currency of every scientific career — the part of training that decides whether a researcher can plan an experiment that survives statistics, read a graph honestly, and reason cleanly under time pressure. Yet in most syllabi it is treated as the residue of school mathematics. We wrote *General Aptitude* to close that gap — and to do so with the rigour, care and respect for the reader that a foundational textbook deserves.

This volume is the General Aptitude component of the IFAS Edutech preparation programme for the *CSIR-NET Life Sciences* examination, but its scope is deliberately broader. It can serve any candidate facing a quantitative-and-reasoning paper — UGC-NET, GATE life sciences, banking and PSU recruitment tests, and the aptitude rounds of competitive postgraduate admissions. Across twenty-six chapters we move from the language of numbers — percentages, ratios, averages — to the machinery of counting, probability and statistics, then through plane and solid mensuration, into the reasoning topics of patterns, codes, directions, syllogisms, calendars and visual puzzles, and finally into the everyday science and applied algebra that the CSIR-NET examiners increasingly favour.

Every chapter follows the same architecture. A short conceptual opening anchors the topic in a real laboratory or fieldwork context. Worked examples are then drawn from our inventory of previous-year questions, with the solution shown in unhurried steps rather than as a one-line formula. A bank of practice MCQs closes the chapter — graded easy, moderate and difficult, each with a full solution and a single sentence on why the wrong options are wrong. A clean two-colour print, with magenta callouts reserved for definitions, key results and the rare didactic warning, separates structure from content at a glance.

The companion mock-test volumes — ten full-length papers built to the latest CSIR-NET pattern, plus per-chapter quick sets — let the reader rehearse under timed conditions and audit their own progress.

We owe sincere thanks to the IFAS Edutech editorial and production teams, to the many faculty colleagues whose marginalia improved the manuscript draft by draft, and most of all to the students whose questions over many teaching cycles shaped the choice of every example in this book.

This is a textbook to be marked, returned to, and argued with. We hope you will use it that way.

Kailash Choudhary
Varun Choudhary

Contents

Chapters

01	The Language of Numbers	01-06
02	Inside the Integers	07-12
03	The Power of Per Cent	13-16
04	Per Cent in the Real World	17-22
05	Buying, Selling and Borrowing	23-27
06	Ratios that Run the World	28-31
07	Mixing and Averaging	32-36
08	Motion, Time and Efforts	37-42
09	The Art of Counting	43-47
10	Chance and Likelihood	48-52
11	Basic Statistics	53-57
12	Charts that Speak	58-62
13	Curves, Slices and Trends	63-67
14	Beyond the Bar Chart	68-72
15	Shapes, Lines and Angles	73-78
16	The Circle Toolbox	79-84
17	Two Circles, Many Stories	85-89
18	Measuring the Flat World	90-94
19	Measuring Solids and Space	95-102
20	Patterns, Codes and Likenesses	103-108
21	Where, Who and How Tall?	109-114
22	Sitting Right, Reasoning Straight	115-120
23	Days, Hours, Cubes and Mirrors	121-126
24	The Puzzle Workshop	127-132
25	Counting Figures in a Composite Diagram	133-140
26	The Aptitude of Everyday Science	141-146

Mock Tests

M01	*Mock Test 1*	147-148
M02	*Mock Test 2*	149-151
M03	*Mock Test 3*	152-154
M04	*Mock Test 4*	155-157
M05	*Mock Test 5*	158-160
M06	*Mock Test 6*	161-163
M07	*Mock Test 7*	164-166
M08	*Mock Test 8*	167-169
M09	*Mock Test 9*	170-172
M10	*Mock Test 10*	173-175

CHAPTER

01 The Language of Numbers

Digits · BODMAS · Cyclicity · Surds and Indices

IN THIS CHAPTER

Numbers are the alphabet of every quantitative subject. This chapter shows you which family each number belongs to, how to combine them in the right order, and three time-savers — unit-digit cycles, surds and indices, and a first peek at logarithms — that reappear in many later chapters. After this chapter you will be able to —

- *name the set (ℕ, 𝕎, ℤ, ℚ, ℝ) of any number you are shown;*
- *use BODMAS to evaluate a long expression without confusion;*
- *predict the last digit of any large power;*
- *simplify surds and apply the four index laws confidently.*

1.1 What is a Number?

A number is a symbol used for three different jobs — to **count** (How many cells in the culture? Two thousand), to **measure** (How concentrated is the buffer? 0.15 M), and to **label** (Take tube number 3). The same ten symbols 0, 1, 2, …, 9 — called **digits** — do all three jobs, because we group them in powers of ten. This is the **decimal system**, and it works in the lab, in the library and in the exam hall.

Most numbers you meet in life science look harmless: 37 (body temperature in °C), 7.4 (blood pH), 6.022×10^{23} (Avogadro's number). They all use the same ten digits but cover very different ‘sizes’, from a fraction to an enormous power of ten. Before working with them in formulae we should know which family of numbers each one belongs to.

WHY TEN DIGITS?

All number systems repeat after a base. Ours is base-10 because we have ten fingers. Computers use base-2 (binary); chemists meet base-16 (hexadecimal) in CAS codes. The arithmetic rules in this chapter work in every base, but every example in this book is base-10.

1.2 The Family of Numbers

Mathematicians group numbers into families. Each family is given a single-letter name in italic-bold so we can talk about them quickly. The families are nested: every member of a smaller family is automatically a member of every larger family that contains it.

Picture the families as a set of nested boxes (Figure 1.1). The innermost box holds the counting numbers you have used since childhood; each surrounding box adds a few new kinds of number — first zero, then negatives, then fractions and finally the irrationals. By the time we reach the outermost box we have every number that can sit on the real number line.

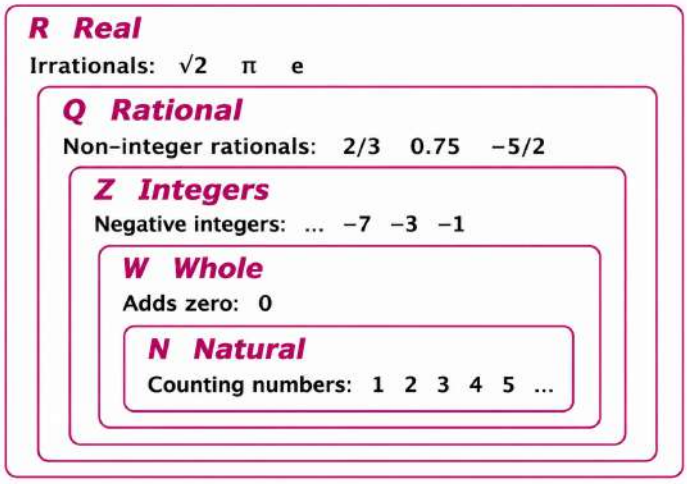


Figure 1.1 Nesting of the five basic number sets — every set contains all sets inside it in this list.

- ***Natural numbers (ℕ)*** — 1, 2, 3, … : the everyday counting numbers; never zero, never negative.
- ***Whole numbers (𝕎)*** — 0, 1, 2, 3, … : the natural numbers together with zero.
- ***Integers (ℤ)*** — …, −2, −1, 0, 1, 2, … : the whole numbers together with their negatives. (Z is from Zahlen — German for 'numbers'.)
- ***Rational numbers (ℚ)*** — any number expressible as $\frac{p}{q}$, where p and q are integers and q ≠ 0; e.g. 3/4, −7, 0.25. (Q is for quotient.)
- ***Irrational numbers*** — cannot be expressed as $\frac{p}{q}$, where p and q are integers and q ≠ 0; their decimal forms neither terminate nor repeat (√2, π, e).
- ***Real numbers (ℝ)*** — rationals together with irrationals; every point on the number line corresponds to a unique real number.

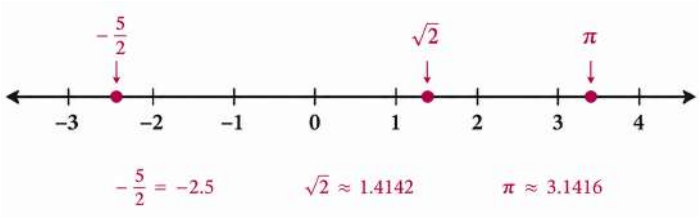


Figure 1.2 The real number line; integers sit at the marked ticks, while √2, π and −5/2 occupy intermediate points.

READING THE NESTING SYMBOL

The symbol ⊂ means 'is contained in'. So ℕ ⊂ 𝕎 reads 'every natural number is also a whole number' — not that the two sets are the same. The full chain ℕ ⊂ 𝕎 ⊂ ℤ ⊂ ℚ ⊂ ℝ simply says: every natural number is whole number, every whole number is an integer, every integer is rational, and every rational is real. Irrational numbers (such as √2 and π) live inside ℝ but outside ℚ.

★ EXAM SHORTCUT

To decide whether a decimal is rational or irrational, look at its tail. If the decimal terminates (0.75) or eventually repeats (0.333…), it is rational. If it never settles into a pattern (1.41421356…), it is irrational.

DID YOU KNOW

The irrationals are uncountably infinite. Georg Cantor proved in 1874 that almost every real number selected at random is irrational — the rationals form only a vanishingly small fraction of the real line.

Within the integers we recognise a few useful sub-families that the examiner asks about often.

- ***Even numbers*** — divisible by 2 (…, −4, −2, 0, 2, 4, …).
- ***Odd numbers*** — not divisible by 2 (…, −3, −1, 1, 3, 5, …).
- ***Prime numbers*** — natural numbers greater than 1 with exactly two divisors, 1 and itself (2, 3, 5, 7, 11, 13, 17, …). Note that 2 is the only even prime.
- ***Composite numbers*** — natural numbers greater than 1 with more than two divisors (4, 6, 8, 9, 10, …). The number 1 is neither prime nor composite.

MEMORISE THESE PRIMES

All primes below 50 — useful for factorisation and divisibility shortcuts: 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47.

1.3 Digit, Place Value and Face Value

A **digit** is one of the ten symbols 0, 1, 2, …, 9. The same digit can have very different 'worth' depending on where it sits in a number.

Consider 4,705:

- The ***face value*** of 7 is just 7 — the symbol itself.
- The ***place value*** of 7 is 7 × 100 = 700, because 7 sits in the hundreds column.

For integers,

$$\text{Place value} = \text{Face value} \times 10^{(\text{position from right} - 1)} \qquad (1.1)$$

The same principle extends to decimal numbers. Digits to the right of the decimal point occupy places worth negative powers of 10: tenths (10^{-1}), hundredths (10^{-2}), thousandths (10^{-3}), and so on. For example, in 0.375, the digit 3 has place value $3 \times 10^{-1} = 0.3$, the digit 7 has place value $7 \times 10^{-2} = 0.07$, and the digit 5 has place value $5 \times 10^{-3} = 0.005$. Thus, moving one place to the right divides the place value by 10, while moving one place to the left multiplies it by 10.

The same idea applies when rounding. To round a decimal to two decimal places we look at the third decimal; if it is 5 or more we round up.

ROUNDING CONVENTION

When the digit immediately after the cut-off is exactly 5 followed by non-zero digits, round up. If it is exactly 5 followed only by zeros, the banker's rule rounds to the nearest even digit; for school arithmetic, the everyday rule simply rounds up.

EXAMPLE 1.1

Rounding off 4.58500001 to the second decimal place gives:

(a) 4.6 (b) 4.58

(c) 4.59 (d) 4.585

Solution. The digit after the second decimal is 5 and the digits behind it (0 0 0 0 1) are not all zero — so we round upward. The 8 in the second decimal place becomes 9, giving 4.59. Answer (c).

1.4 BODMAS — Order of Operations

When several operations appear in a single expression, **BODMAS** tells us which runs first. Within the same priority level (× and ÷, or + and −) we evaluate from left to right.

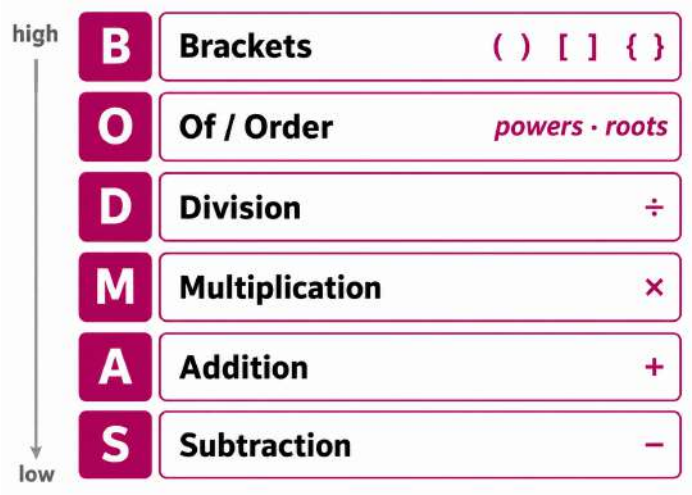


Figure 1.3 The BODMAS hierarchy — operations higher in the stack are applied first.

KEY POINT

Brackets are always evaluated first; a nested inner bracket is solved before its outer one. The word 'of' behaves like multiplication but ranks immediately above × and ÷.

EXAMPLE 1.2

Dividing sixty by half and adding half of forty yields:

(a) 35 (b) 50

(c) 70 (d) 140

Solution. 'Dividing 60 by half' means 60 ÷ (1/2) = 120, not 60/2. Then add half of 40, namely 20. Hence 120 + 20 = 140. Answer (d).

EXAMPLE 1.3

Simplify $16 - 4 \times 3 + 2$ of $6 \div 4$.

(a) 7 (b) 9

(c) 11 (d) 13

Solution. Apply BODMAS in order. '2 of 6' = 12; 12 ÷ 4 = 3. Now the expression reads 16 − 4 × 3 + 3 = 16 − 12 + 3 = 7. Answer (a).

★ EXAM SHORTCUT

When the question contains 'of', mentally replace it by 'times, but earlier'. The phrase 'a of b' acts like multiplication and is settled before any nearby × or ÷ symbol.

1.5 Unit-Digit Cyclicity

Questions of the form 'what is the **last digit** of 2 raised to a large power?' are common because the full power cannot be computed by hand. The unit digit follows a **short cycle**, and knowing the cycle solves the question in seconds.

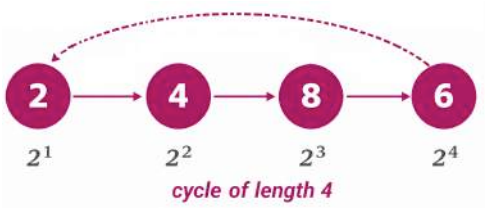


Figure 1.4 Powers of 2 cycle through the unit digits 2, 4, 8, 6.

- 0, 1, 5, 6 — cycle length 1 (powers always end in 0, 1, 5, 6 respectively).
- 4, 9 — cycle length 2 (4: 4, 6, 4, 6, … 9: 9, 1, 9, 1, …).
- 2, 3, 7, 8 — cycle length 4.

$$\text{unit digit } (a^n) = \text{unit digit } (a^{n \bmod L}) \quad (1.2)$$

where L = cycle length

EXAMPLE 1.4

What is the digit at the unit's place of 7^{2025}?

(a) 1 (b) 3

(c) 7 (d) 9

Solution. The cycle of 7 has length 4: 7, 9, 3, 1, 7, … . Since 2025 mod 4 = 1, the unit digit matches that of 7 raised to the first power, which is 7. **Answer (c).**

NOTE

If n mod L equals 0, use L itself (not 0) as the working exponent. For example, the unit digit of 2 raised to the 8th power is found from 2 raised to the 4th = 16, giving 6.

★ EXAM SHORTCUT

Ignore everything except the unit digit of the base. The unit digit of 2974^{301} is identical to that of 4^{301} — because every other digit of the base disappears under multiplication of only the last column.

1.6 Surds and Indices

A **surd** is a root that cannot be reduced to a rational number (for example $\sqrt{2}$ or $\sqrt[3]{5}$). An **index**, or exponent, is the superscript that tells how many times a base multiplies itself ($2^3 = 2 \times 2 \times 2 = 8$).

$$a^m \cdot a^n = a^{m+n}$$

$$a^m / a^n = a^{m-n}$$

$$(a^m)^n = a^{mn} \quad (1.3)$$

$$a^0 = 1 \quad (a \neq 0)$$

$$a^{-n} = 1 / a^n$$

$$a^{1/n} = \sqrt[n]{a} \quad (1.4)$$

COMMON SURD IDENTITIES

$\sqrt{50} = \sqrt{(25 \cdot 2)} = 5\sqrt{2}$. $\sqrt{(a^2 b)} = a\sqrt{b}$. To rationalise the denominator, multiply by the appropriate surd; e.g. $1/\sqrt{3} = \sqrt{3}/3$.

EXAMPLE 1.5

The geometric mean of the squares of two positive integers is 10. The smallest possible sum of these two integers is:

(a) 7 (b) 8

(c) 10 (d) 11

Solution. GM$(a^2, b^2) = \sqrt{(a^2 b^2)} = ab = 10$. Integer pairs with ab = 10 are (1, 10) and (2, 5), with sums 11 and 7 respectively. The smallest is 7. **Answer (a).**

1.7 A First Look at Logarithms

If $a^x = N$ we write $x = \log_a N$ and read it 'the logarithm of N to base a'. Logarithms convert multiplication into addition — a habit life sciences already exploit in pH, in sound-intensity decibels, and in absorbance spectroscopy.

$$\log (ab) = \log a + \log b$$

$$\log (a / b) = \log a - \log b$$

$$\log (a^n) = n \cdot \log a \quad (1.5)$$

SCIENTIFIC NOTATION

A number written as $a \times 10^n$, with $1 \le a < 10$, is in scientific notation. Examples: 6.022×10^{23} (Avogadro), 1.66×10^{-27} kg (atomic mass unit). Multiplying these is easy: add the exponents ($10^m \cdot 10^n = 10^{m+n}$).

DID YOU KNOW

The familiar definition $pH = -\log_{10} [H^+]$ is just a logarithm renamed: pure water with $[H^+] = 10^{-7}$ mol/L therefore has pH 7. A jump of one pH unit means a ten-fold change in hydrogen-ion concentration.

★ EXAM SHORTCUT

$\log_{10} 2 \approx 0.301$ and $\log_{10} 3 \approx 0.477$. With just these two values you can estimate log of any number whose prime factors are 2, 3 and 5 (since log 5 = log 10 − log 2 = 0.699). Worth memorising.

1.8 Sequences, Sums and a Quick Algebra Toolkit

Many counting and arithmetic shortcuts in later chapters depend on the same handful of results. We close Chapter 1 with the three sum formulas every student should know by heart, the rules for arithmetic and geometric sequences, the modulus of a real number, and the quadratic formula. None of them is hard; they just save a great deal of time in the exam hall.

- ***Sum of the first n natural numbers*** — the result Gauss famously discovered as a schoolboy by pairing 1 + n, 2 + (n − 1), … each pair summing to n + 1.

$$1 + 2 + 3 + \cdots + n = \frac{n(n+1)}{2} \quad (1.6)$$

- ***Sum of the squares*** of the first n natural numbers.

$$1^2 + 2^2 + \cdots + n^2 = \frac{n(n+1)(2n+1)}{6} \quad (1.7)$$

- ***Sum of the cubes*** — equals the square of the simple sum.

$$1^3 + 2^3 + \cdots + n^3 = \left[\frac{n(n+1)}{2}\right]^2 \quad (1.8)$$

An **arithmetic progression (AP)** is a sequence whose successive terms differ by a constant d. The first term is a, the n-th term is $a_n = a + (n-1)d$, and the sum of the first n terms is $S_n = \frac{n}{2}(2a + (n-1)d) = \frac{n}{2}(a + l)$, where l is the last term.

A **geometric progression (GP)** is one in which each term is r times the previous. The n-th term is $a_n = a \cdot r^{n-1}$, and the sum of the first n terms is $S_n = \frac{a(1-r^n)}{1-r}$ for $r \neq 1$. If $|r| < 1$ the infinite GP converges to $\frac{a}{1-r}$ — the foundation of every geometric-decay argument used later in the book.

MODULUS OF A REAL NUMBER

$|x|$ is the size of x without its sign — so $|3| = |-3| = 3$. Two useful identities: $|x \cdot y| = |x| \cdot |y|$ and the triangle inequality $|x + y| \leq |x| + |y|$.

QUADRATIC FORMULA

The roots of $ax^2 + bx + c = 0$ are $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$.

The discriminant $D = b^2 - 4ac$ decides the type: $D > 0$ gives two distinct real roots, $D = 0$ gives one repeated root, and $D < 0$ gives a pair of complex conjugates.

★ EXAM SHORTCUT

Whenever a exam question asks for 'the sum of the first n …', 'the n-th term of …' or 'the dots in row n of a pyramid', reach straight for one of (1.6)–(1.8) above. Recognising the pattern is half the work.

Quick Recap

- Number sets nest: every natural is whole, every whole is integer, every integer is rational, and every rational is real — i.e. $\mathbb{N} \subset \mathbb{W} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$. Irrational numbers live inside $\mathbb{R}$ but outside $\mathbb{Q}$.
- Place value = face value × 10 raised to (position − 1). For decimal numbers, moving one place to the right of decimal divides the place value by 10, while moving one place to the left multiplies it by 10.
- BODMAS order: Brackets → Of/Order → ÷ → × → + → −. Within the same priority, work left to right.
- Unit-digit cycles — length 1 (0, 1, 5, 6); length 2 (4, 9); length 4 (2, 3, 7, 8).
- Index laws: $a^m \cdot a^n = a^{m+n}$; $(a^m)^n = a^{mn}$; $a^{-n} = 1/a^n$.
- Logarithm: log(ab) = log a + log b. Familiar example: pH = $-\log_{10}[H^+]$.
- Sum identities: $1+2+\ldots+n = n(n+1)/2$; $1^2+2^2+\ldots+n^2 = n(n+1)(2n+1)/6$; $1^3+2^3+\ldots+n^3 = [n(n+1)/2]^2$.
- AP n-th term $a_n = a + (n-1)d$; GP n-th term $a_n = a \cdot r^{n-1}$; infinite GP ($|r|<1$) sums to $a/(1-r)$.
- $|x|$ = magnitude of x; quadratic roots = $(-b \pm \sqrt{(b^2-4ac)})/(2a)$.

Practice

Attempt all 15 questions within 15 minutes.

1. The digit at the unit's place of 3 raised to the 100th power is
   (a) 1 (b) 3
   (c) 7 (d) 9
2. Which of the following is irrational?
   (a) 22/7 (b) $\sqrt{16}$
   (c) $\sqrt{7}$ (d) 0.333…

3. Simplify $60 \div 5$ of $2 + 4 \times 3 - 7$.
   (a) 8 (b) 11
   (c) 12 (d) 17

4. If 2 raised to x equals 1/32, then x equals
   (a) 5 (b) −5
   (c) 1/5 (d) −1/5

5. The place value of 8 in 38 904 is
   (a) 8 (b) 800
   (c) 8 000 (d) 80 000

6. How many zeros does the product $5 \times 10 \times 15 \times 20 \times 25 \times 30$ end with?
   (a) 3 (b) 4
   (c) 5 (d) 6

7. The value of $(\sqrt{3} + 1)(\sqrt{3} - 1)$ is
   (a) 1 (b) 2
   (c) 3 (d) 4

8. $\log_{10} 2 + \log_{10} 5$ equals
   (a) 0 (b) 1
   (c) 5 (d) $\log_{10} 7$

9. The smallest positive integer that leaves remainder 1 when divided by 2, 3, 4 and 5 is
   (a) 31 (b) 41
   (c) 61 (d) 121

10. If a raised to (1/3) equals 2, then a^2 equals
    (a) 8 (b) 16
    (c) 32 (d) 64

11. Rounded to two decimal places, 0.49500001 becomes
    (a) 0.49 (b) 0.50
    (c) 0.495 (d) 0.500

12. The number of positive factors of 60 is
    (a) 8 (b) 10
    (c) 12 (d) 16

13. The value of $\sqrt{50}$ lies between
    (a) 6 and 7 (b) 7 and 8
    (c) 8 and 9 (d) 9 and 10

14. If p and q are two distinct primes both greater than 5, the unit digit of $p \cdot q$ can NEVER be
    (a) 1 (b) 3
    (c) 5 (d) 9

15. Simplify $(a^4 b^{-2}) \div (a^{-1} b^3)$.
    (a) $a^3 b^5$ (b) $a^5 b^5$
    (c) $a^5 b^{-5}$ (d) $a^3 b^{-5}$

Solutions

1. The cycle of unit digits of powers of 3 is (3, 9, 7, 1), of length 4. Since $100 \equiv 0 \pmod 4$, we use the 4th term of the cycle, which is 1. **Answer (a).**

2. 22/7 is a ratio of integers; $\sqrt{16} = 4$; $0.333\ldots = 1/3$ are all rational. Only $\sqrt{7}$ is irrational because 7 is not a perfect square. **Answer (c).**

3. Apply BODMAS. '5 of 2' is treated as multiplication and evaluated first inside the division step: $60 \div (5 \times 2) = 60 \div 10 = 6$. Now $4 \times 3 = 12$. Combining, $6 + 12 - 7 = 11$. **Answer (b).**

4. $1/32 = 1 / 2^5 = 2^{-5}$, so $x = -5$. **Answer (b).**

5. In 38 904 the digit 8 sits in the thousands position. Place value $= 8 \times 10^3 = 8\,000$. **Answer (c).**

6. Trailing zeros equal the number of (2, 5) factor pairs. Counting 5s in the factors: $5(1) + 10(1) + 15(1) + 20(1) + 25(2) + 30(1) = 7$. Counting 2s: $5(0) + 10(1) + 15(0) + 20(2) + 25(0) + 30(1) = 4$. Minimum is 4. **Answer (b).**

7. Difference of squares: $(\sqrt{3})^2 - (1)^2 = 3 - 1 = 2$. **Answer (b).**

8. $\log a + \log b = \log(ab)$; hence $\log_{10} 2 + \log_{10} 5 = \log_{10} 10 = 1$. **Answer (b).**

9. Let N be the required integer. Then $N - 1$ must be a common multiple of 2, 3, 4 and 5; the smallest such multiple is LCM(2, 3, 4, 5) = 60. So $N = 60 + 1 = 61$. **Answer (c).**

10. Cube both sides of the given relation: $a = 2^3 = 8$. Therefore $a^2 = 8^2 = 64$. **Answer (d).**

11. The third decimal is 5 and the digits behind it (0 0 0 0 1) are not all zero, so we round upward. $0.49 \rightarrow 0.50$. **Answer (b).**

12. $60 = 2^2 \cdot 3^1 \cdot 5^1$. Total number of divisors $= (2 + 1)(1 + 1)(1 + 1) = 12$. **Answer (c).**

13. $7^2 = 49$ and $8^2 = 64$. Since $49 < 50 < 64$, we have $7 < \sqrt{50} < 8$. **Answer (b).**

14. Every prime greater than 5 ends in 1, 3, 7 or 9. Products of two such unit digits give unit digits drawn from {1, 3, 7, 9} only. The digit 5 is therefore impossible. **Answer (c).**

15. Subtract exponents separately. For the a-terms: $4 - (-1) = 5$, giving a^5. For the b-terms: $-2 - 3 = -5$, giving b^{-5}. **Answer (c).**

CHAPTER

02 Inside the Integers

Divisibility · Remainders · Factors · HCF and LCM

IN THIS CHAPTER

We zoom in on the integers — the everyday whole-number side of mathematics — and meet the tools that examiners reach for again and again: quick divisibility tests, remainder thinking, factorisation, and the twin ideas of highest common factor (HCF) and lowest common multiple (LCM). After this chapter you will be able to —

- *decide in a glance whether a number is divisible by 2, 3, 4, 5, 6, 8, 9 or 11;*
- *find the remainder when one integer is divided by another, without long division;*
- *factorise any integer into primes and use that to compute HCF and LCM;*
- *spot the digit-and-place tricks the examiner uses with two-digit numbers and perfect squares.*

2.1 Divisibility Rules — The Nine Quick Tests

An integer a is said to be **divisible by** another integer b (with b ≠ 0) if the division a ÷ b leaves no remainder — that is, a = b × q for some integer q. We write **b ∣ a** and read it 'b divides a'.

Long division is slow. The nine rules in Figure 2.1 let you test divisibility in seconds and are worth memorising for the exam hall.

- ***For 7*** — the standard trick is: take the last digit, double it, and subtract it from the number formed by the remaining digits. If the result is divisible by 7 (or zero), so is the original. Example: 1274 → 127 − 8 = 119 → 11 − 18 = −7, divisible by 7.
- ***For 13*** — multiply the last digit by 4 and add to the rest; iterate until small. Less common in the exam.

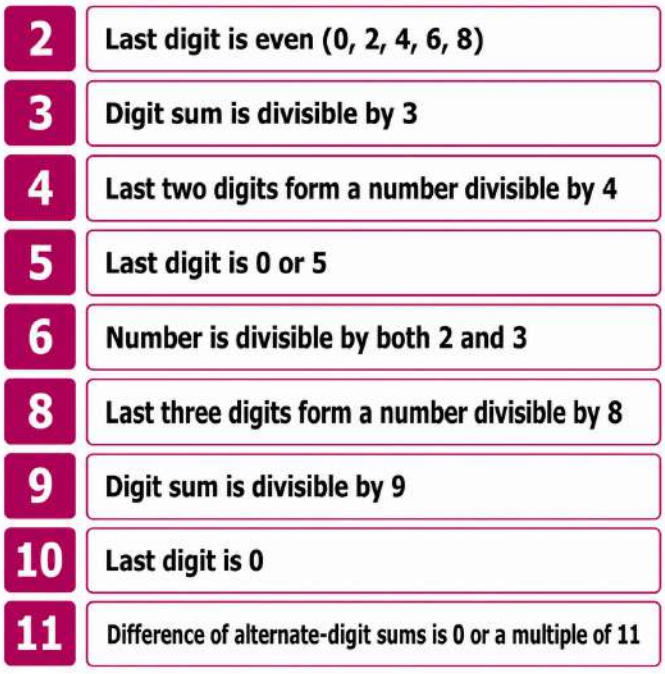


Figure 2.1 Quick divisibility tests for 2, 3, 4, 5, 6, 8, 9, 10 and 11.

KEY POINT

A number is divisible by 6 only if it passes both the 2-test and the 3-test simultaneously. Likewise, divisible by 12 means divisible by both 3 and 4; by 15 means by both 3 and 5; by 18 means by both 2 and 9.

★ EXAM SHORTCUT

If you are asked whether a digit-sum trick applies to numbers other than 3 and 9, the answer is no for 6, 4, 8 — those rules look at the last few digits, not the digit-sum.

EXAMPLE 2.1

From a two-digit number, the sum of its digits is subtracted. The resulting number is:

(a) always divisible by 6 (b) always divisible by 9

(c) always divisible by 11 (d) always a perfect square

Solution. Let the number be 10a + b where a is the tens digit and b the units digit. Subtracting the digit sum a + b gives (10a + b) − (a + b) = 9a — always a multiple of 9. Answer (b).

2.2 Remainders and Modular Thinking

When a is divided by n, we get a quotient q and a remainder r such that —

$$a = n \cdot q + r, \quad 0 \le r < n \qquad (2.1)$$

We write $a \equiv r \pmod{n}$ and read it 'a is congruent to r modulo n'. Two integers are congruent mod n if they differ by a multiple of n.

- Examples: $17 = 5 \cdot 3 + 2$, so $17 \equiv 2 \pmod 5$.
- $100 \equiv 0 \pmod 4$, $100 \equiv 1 \pmod 9$, $100 \equiv 2 \pmod 7$.

USEFUL CONGRUENCE RULES

If $a \equiv r \pmod n$ and $b \equiv s \pmod n$, then $a + b \equiv r + s$, $a - b \equiv r - s$, and $a \cdot b \equiv r \cdot s$, all mod n. So we can replace a number by its remainder before adding or multiplying.

★ EXAM SHORTCUT

'A number when increased by k becomes divisible by m' means the number is $\equiv -k \pmod m$, or equivalently $\equiv (m - k) \pmod m$. The remainder is read off directly — no long division needed.

EXAMPLE 2.2

A number when increased by 17 becomes divisible by 392. What is the remainder when the number is divided by 98?

(a) 49 (b) 75

(c) 81 (d) 17

Solution. Let the number be N. Then N + 17 = 392 k for some integer k, so N = 392 k − 17. Since 392 = 4 · 98, the term 392 k is divisible by 98, leaving $N \equiv -17 \pmod{98}$. Adding 98 to bring the remainder into [0, 98), we get 98 − 17 = 81. Answer (c).

2.3 Factors, Multiples and Prime Factorisation

A **factor** of n is any positive integer that divides n exactly. The factors of 12 are 1, 2, 3, 4, 6 and 12 — six in total. A **multiple** of n is any number n itself, 2n, 3n, … . Every integer greater than 1 can be written, in essentially one way, as a product of primes — this is the **fundamental theorem of arithmetic**.

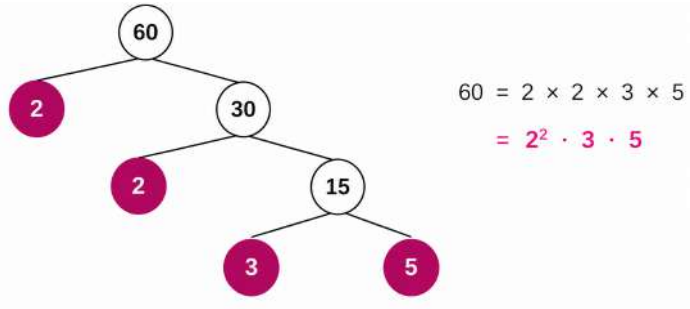


Figure 2.2 Factor tree of 60 — primes circled in magenta; the right side is the unique prime factorisation.

If $n = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$, then the total number of positive factors of n is

$$(a_1 + 1)(a_2 + 1) \cdots (a_k + 1) \qquad (2.2)$$

- Example: $60 = 2^2 \cdot 3 \cdot 5$, number of factors = (2 + 1)(1 + 1)(1 + 1) = 12.

SUM AND PRODUCT OF FACTORS

Sum of factors of $n = (1 + p_1 + p_1^2 + \ldots + p_1^{a1})(1 + p_2 + \ldots + p_2^{a2}) \cdot \ldots \cdot$ Product of all factors of n equals n raised to the power (number of factors / 2).

EXAMPLE 2.3

The sum of two positive integers is 14. Then their product CANNOT be divisible by:

(a) 12 (b) 13

(c) 14 (d) 49

Solution. The seven possible products are 13, 24, 33, 40, 45, 48, 49 — from the pairs (1, 13), (2, 12), (3, 11), (4, 10), (5, 9), (6, 8) and (7, 7). 12 divides 24 and 48; 13 divides 13; 49 divides 49. But 14 = 2 · 7 requires both a factor of 2 and a factor of 7 in the product, and no listed product has both — so 14 cannot divide any of them. Answer (c).

2.4 HCF and LCM

The **highest common factor (HCF)** of two integers is the largest integer that divides both. The **lowest common multiple (LCM)** is the smallest positive integer that both divide. From the prime factorisations, the recipe is short.

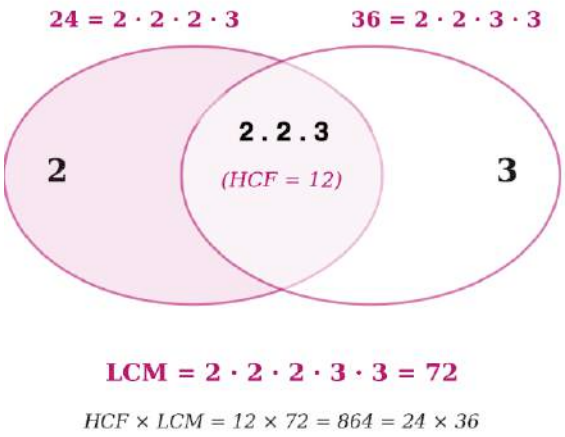


Figure 2.3 HCF is the intersection of the prime-factor lists; LCM is the union (counting each prime to its higher power).

$$\text{HCF} \cdot \text{LCM} = a \cdot b \quad \text{(for two integers } a, b\text{)} \tag{2.3}$$

- ***HCF*** — take every prime that appears in both factorisations to its lower power, then multiply.
- ***LCM*** — take every prime appearing in either factorisation to its higher power, then multiply.
- Two integers are ***co-prime*** (no common factor other than 1) when their HCF is 1. Co-prime pairs have LCM equal to their product.

PERIODIC EVENTS

Any question about events that repeat together — clocks ticking, lights flashing, traffic lights cycling — is an LCM question. 'When do they next coincide?' asks the LCM of the individual periods.

★ EXAM SHORTCUT

If the question asks 'the smallest number that leaves remainder r when divided by each of p, q, s', the answer is LCM(p, q, s) + r. The remainder is the same, so the difference between the answer and r is divisible by all of them, forcing it to be a common multiple.

EXAMPLE 2.4

Four clocks ring at different intervals: 30, 60, 80 and 90 minutes. If they all rang together at 10 AM, when do they next ring together?

(a) 1 PM (b) 2 PM
(c) 10 PM (d) Next day 4 AM

Solution. The next coincidence happens after LCM(30, 60, 80, 90) minutes. Factorise — $30 = 2\cdot3\cdot5$, $60 = 2^2\cdot3\cdot5$, $80 = 2^4\cdot5$, $90 = 2\cdot3^2\cdot5$ — and take the highest power of each prime: $2^4 \cdot 3^2 \cdot 5 = 720$ minutes = 12 hours. Twelve hours after 10 AM is 10 PM. Answer (c).

2.5 Primes and Their Habits

A **prime** is a natural number greater than 1 with exactly two divisors, 1 and itself. The primes below 50 are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47 — fifteen of them, worth memorising. Figure 2.4 visualises the same list as a Sieve of Eratosthenes.

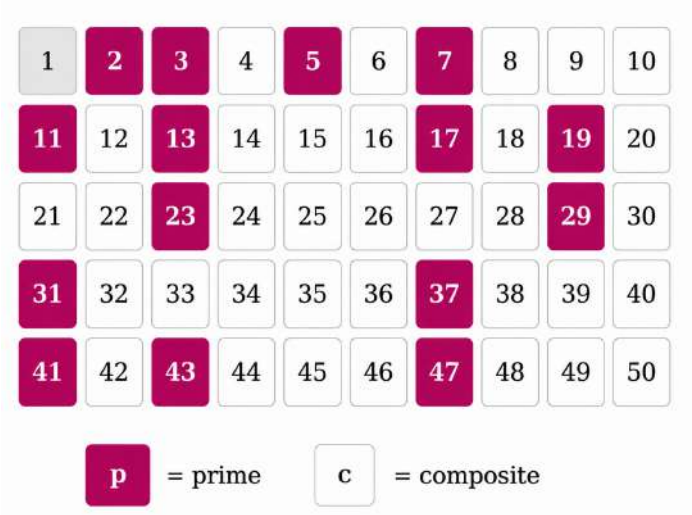


Figure 2.4 Numbers 1–50 with primes highlighted in magenta (the Sieve of Eratosthenes).

- 2 is the only even prime — all other primes are odd.
- Every prime greater than 3 has the form $6k \pm 1$ (i.e. $6k + 1$ or $6k - 1$).
- Every prime greater than 5 has its last digit in the set {1, 3, 7, 9}.
- There are infinitely many primes — proven by Euclid c. 300 BC.
- A ***perfect number*** equals the sum of all its proper divisors. The first four are $6 = 1 + 2 + 3$, $28 = 1 + 2 + 4 + 7 + 14$, 496 and 8128. Every known perfect number is even; whether any odd one exists is still unsolved.

TWIN PRIMES

Two primes that differ by 2 are called twin primes — (3, 5), (5, 7), (11, 13), (17, 19), (29, 31), (41, 43) …. Whether infinitely many twin-prime pairs exist remains one of the great open problems of mathematics.

DID YOU KNOW

Every known even perfect number has the form $2^{(p-1)} \times (2^p - 1)$, where $2^p - 1$ is itself prime — a Mersenne prime.

EXAMPLE 2.5

For any three distinct prime numbers, all greater than 13, which of the following is NOT true?

(a) Their sum is even

(b) Their sum is greater than 41

(c) Their product is odd

(d) Their product is greater than 5000

Solution. All primes greater than 13 are odd. The sum of three odd numbers is itself odd, never even — so option (a) is false. The smallest three such primes are 17, 19, 23, giving sum 59 (> 41) and product 7429 (> 5000), so (b), (c) and (d) are all true. Only (a) is the wrong statement. Answer (a).

2.6 Two-Digit Number Tricks

A two-digit number with tens digit a and units digit b is written 10a + b. Reversing the digits gives 10b + a. The relationships below recur in exam questions.

$$(10a + b) + (10b + a) = 11\,(a + b) \quad (2.4)$$

$$(10a + b) - (10b + a) = 9\,(a - b) \quad (2.5)$$

$$(10a + b)^2 - (10b + a)^2 = 99\,(a^2 - b^2) \quad (2.6)$$

TWO CONSEQUENCES

(i) The sum of any two-digit number and its reverse is always a multiple of 11. (ii) The difference is always a multiple of 9. (iii) The difference of their squares is always a multiple of 99.

EXAMPLE 2.6

The difference between two two-digit numbers, one formed by reversing the digits of the other, is 27. The difference between the two digits is:

(a) 1 (b) 2

(c) 3 (d) 4

Solution. By (2.5), the difference equals 9(a − b), so 9(a − b) = 27 → a − b = 3. Answer (c).

★ EXAM SHORTCUT

When a digit-reverse questions asks 'what is the digit difference / digit sum?', divide the given answer by 9 (for difference) or 11 (for sum). The answer is one-step.

2.7 Perfect Squares and Sums of Squares

A **perfect square** is the square of an integer (0, 1, 4, 9, 16, 25 …). They have several fingerprints — handy for spotting trick options in exam questions.

- The last digit of a perfect square is in the set {0, 1, 4, 5, 6, 9} — never 2, 3, 7 or 8.
- A perfect square leaves remainder 0 or 1 when divided by 4 — never 2 or 3.
- A perfect square has an odd number of factors.
- Sum of two squares — an integer can be written as $a^2 + b^2$ only if every prime of the form 4k + 3 in its factorisation appears to an even power (Fermat's theorem on sums of two squares).
- Sum of three squares — Legendre's theorem: a positive integer is expressible as a sum of three squares unless it has the form $4^m(8n + 7)$. The integer 7 is the first counter-example (7, 15, 23, 28, 31 …).

LAGRANGE FOUR-SQUARE THEOREM

Every positive integer can be written as the sum of at most four squares — a result Lagrange proved in 1770. So the cut-off is sharp: three squares may fail (for 7, 15, 23, …), but four always succeed.

EXAMPLE 2.7

Which among the following integers can NEVER be written as the sum of squares of three integers?

(a) 6 (b) 7

(c) 8 (d) 9

Solution. By Legendre's three-square theorem, the integers that cannot be expressed as a sum of three squares are those of the form $4^m(8n + 7)$. For n = 0, m = 0 the smallest is 7. Hence 7 is the only one of the four options that fails — 6 = 1 + 1 + 4, 8 = 4 + 4 + 0, 9 = 9 + 0 + 0. Answer (b).

2.8 Powers of a Prime in a Product — Counting Trailing Zeros

Trailing zeros come from pairs of factors (2, 5). Since 2 is plentiful in any factorial or long product, the number of trailing zeros is fixed by the number of 5s available.

$$\text{Number of trailing zeros of } n! = \lfloor n/5 \rfloor + \lfloor n/25 \rfloor + \lfloor n/125 \rfloor + \ldots \quad (2.7)$$

For a general product such as $5 \cdot 10 \cdot 15 \cdot \ldots \cdot 30$ we count factors of 5 and factors of 2 separately; the smaller count is the number of trailing zeros.

EXAMPLE 2.8

The product $5 \times 10 \times 15 \times 20 \times 25 \times 30$ ends with how many zeros?

(a) 3 (b) 4
(c) 5 (d) 7

Solution. Count the factors of 5: 5 (one) + 10 (one) + 15 (one) + 20 (one) + 25 (two) + 30 (one) = 7. Count the factors of 2: 10 (one) + 20 (two) + 30 (one) = 4. The smaller count, 4, gives the number of trailing zeros. Answer (b).

EXAMPLE 2.9

If 9 papers, printed on both sides, are drawn from a book, the sum of their page numbers CAN NEVER be:

(a) 171 (b) 183
(c) 195 (d) 200

Solution. Sheet k carries pages $2k - 1$ and $2k$, summing to $4k - 1$. Nine sheets give a total of $4(k_1 + k_2 + \dots + k_9) - 9 = 4S - 9$ where S is the sum of the nine sheet indices. So the total must satisfy total $\equiv -9 \equiv 3 \pmod 4$. Check each option: $171 = 4 \cdot 45 - 9$ ✓, $183 = 4 \cdot 48 - 9$ ✓, $195 = 4 \cdot 51 - 9$ ✓, but 200 needs $S = 52.25$ — not an integer. So 200 is impossible. Answer (d).

★ EXAM SHORTCUT

Highest power of a prime p in n! is computed by Legendre's formula $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \dots$ (stop when the term is zero). Worth memorising — it covers every trailing-zero exam questions and every 'largest power of 3 in 50!' style question.

DID YOU KNOW

Goldbach's conjecture, posed in 1742, says every even integer greater than 2 is the sum of two primes. It has been verified by computer for all even numbers up to 4×10^{18} — but a general proof is still missing after almost three centuries.

Quick Recap

- A | B means A divides B exactly; rules for 2, 3, 4, 5, 6, 8, 9, 10 and 11 give instant tests.
- $a \equiv r \pmod n$ means $a - r$ is a multiple of n; 'increased by k becomes divisible by m' → remainder is $m - k$.
- If $n = p_1^{a1} \cdot p_2^{a2} \cdot \dots$ then number of factors = $(a_1 + 1)(a_2 + 1) \cdot \dots$
- HCF × LCM = a × b; LCM solves all periodic-events questions.
- Two-digit reversal: $(10a + b) \pm (10b + a)$ is a multiple of 11 or 9 respectively.
- Perfect squares end in 0, 1, 4, 5, 6, 9 only; integers of the form $4^m(8n + 7)$ — e.g. 7, 15, 23 — cannot be written as a sum of three squares.
- Highest power of prime p in n! is $\lfloor n/p \rfloor + \lfloor n/p^2 \rfloor + \lfloor n/p^3 \rfloor + \dots$ — drives every trailing-zero question.

Practice

Attempt all 15 questions within 15 minutes.

1. Number of trailing zeros in 25! is
   (a) 5 (b) 6
   (c) 7 (d) 8
2. Which of the following is divisible by 11?
   (a) 1 257 (b) 2 233
   (c) 9 876 (d) 5 432
3. The HCF of 84 and 126 is
   (a) 14 (b) 21
   (c) 28 (d) 42
4. Three bells toll at intervals of 9, 12 and 15 minutes. After how many minutes will they all toll together?
   (a) 120 (b) 150
   (c) 180 (d) 240
5. The largest power of 3 in 50! is
   (a) 20 (b) 22
   (c) 24 (d) 27
6. A number leaves remainder 4 when divided by 7. What is the remainder when twice the number is divided by 7?
   (a) 1 (b) 4
   (c) 6 (d) 8
7. Which of the integers below CANNOT be written as the sum of squares of three integers?
   (a) 10 (b) 12
   (c) 15 (d) 18
8. The sum of a two-digit number and the number obtained by reversing its digits is 121. The sum of its digits is
   (a) 7 (b) 9
   (c) 11 (d) 13

9. The difference of the squares of two distinct two-digit numbers, one being the reverse of the other, is always divisible by
   (a) 7 (b) 9
   (c) 11 (d) 99

10. How many of the integers from 1 to 100 are divisible by 6?
   (a) 12 (b) 14
   (c) 16 (d) 18

11. Which of the following is the smallest sum of three distinct primes each greater than 10?
   (a) 33 (b) 41
   (c) 45 (d) 47

12. Which of the following CAN be a perfect square?
   (a) 1 098 (b) 2 022
   (c) 3 600 (d) 4 712

13. From a three-digit number, the sum of its digits is subtracted. The result is always divisible by
   (a) 3 (b) 6
   (c) 9 (d) 11

14. A book has pages numbered 1 to 100. The sum of all the page numbers is
   (a) 4 950 (b) 5 000
   (c) 5 050 (d) 5 100

15. The smallest positive integer that leaves remainder 3 when divided by 5, 6 and 9 is
   (a) 33 (b) 45
   (c) 93 (d) 183

Solutions

1. Use Legendre's formula: $\lfloor 25/5 \rfloor + \lfloor 25/25 \rfloor = 5 + 1 = 6$. Answer (b).

2. The 11-test: difference of sums of alternate digits must be 0 or a multiple of 11. For 2 233: $(2 + 3) - (2 + 3) = 0$, divisible. For 1 257: $1 + 5 - 2 - 7 = -3$ (no); 9 876: $9 + 7 - 8 - 6 = 2$ (no); 5 432: $5 + 3 - 4 - 2 = 2$ (no). Only 2 233 passes. Answer (b).

3. $84 = 2^2 \cdot 3 \cdot 7$; $126 = 2 \cdot 3^2 \cdot 7$. Common primes to lower power: $2 \cdot 3 \cdot 7 = 42$. Answer (d).

4. $\mathrm{LCM}(9, 12, 15) = \mathrm{LCM}(3^2, 2^2 \cdot 3, 3 \cdot 5) = 2^2 \cdot 3^2 \cdot 5 = 180$ minutes. Answer (c).

5. Legendre: $\lfloor 50/3 \rfloor + \lfloor 50/9 \rfloor + \lfloor 50/27 \rfloor = 16 + 5 + 1 = 22$. Answer (b).

6. If $N \equiv 4 \pmod 7$, then $2N \equiv 8 \equiv 1 \pmod 7$. Answer (a).

7. $15 = 4^0 \cdot (8 \cdot 1 + 7)$, of Legendre's exclusion form $4^m(8n + 7)$. Others: $10 = 9 + 1 + 0$; $12 = 4 + 4 + 4$; $18 = 16 + 1 + 1$. Answer (c).

8. By (2.4) the sum is $11(a + b)$. So $11(a + b) = 121 \rightarrow a + b = 11$. Answer (c).

9. From (2.6) the difference equals $99\,(a^2 - b^2)$, so always divisible by 99. Answer (d).

10. $\lfloor 100/6 \rfloor = 16$. Answer (c).

11. Three smallest distinct primes greater than 10 are 11, 13 and 17; their sum is 41. Answer (b).

12. Last digits 8, 2 and 2 disqualify (a), (b), (d). $3\,600 = 60^2$, a perfect square. Answer (c).

13. If $N = 100\,a + 10\,b + c$ then $N - (a + b + c) = 99\,a + 9\,b = 9(11\,a + b)$ — always a multiple of 9. Answer (c).

14. Sum $1 + 2 + \ldots + n = n(n + 1)/2 = 100 \cdot 101/2 = 5\,050$. Answer (c).

15. Required $N - 3$ = common multiple of 5, 6, 9 $\rightarrow$ LCM = 90. So $N = 90 + 3 = 93$. Answer (c).

CHAPTER

03 The Power of Per Cent

Conversions · Successive change · Reverse % · Error propagation

IN THIS CHAPTER

Per cent is the universal language of comparison — pH, OD, concentration, marks, market share. In this chapter we learn how to move quickly between fractions, decimals and per cent, how to handle one or several successive changes (and why two equal-sized changes in opposite directions don't cancel), and how a small per cent error in a measured quantity becomes a much larger error in any derived quantity. After this chapter you will be able to —

- *convert any fraction, decimal or ratio to and from a percentage in one mental step;*
- *compute the net effect of two or more successive percentage changes;*
- *solve reverse percentage problems (find the original from the final);*
- *propagate a per cent error from a measured quantity to a derived one;*
- *apply the mixture-replacement formula to dilution and substitution questions.*

3.1 What is a Per Cent?

The word **per cent** comes from the Latin per centum — '**for each hundred**'. So 7 per cent means seven parts out of every hundred. The symbol % is itself a stylised '/100'.

Because per cent is just a fraction in disguise, every per-cent question can also be solved as a fraction or a decimal. Choose whichever feels lightest in the moment.

FRACTIONS WORTH MEMORISING

1/2 = 50% 1/3 ≈ 33.3% 1/4 = 25% 1/5 = 20% 1/8 = 12.5% 2/3 ≈ 66.7% 3/4 = 75% 3/5 = 60% 1/6 ≈ 16.7% 1/10 = 10% 1/20 = 5% 1/25 = 4%. Recognising these on sight saves long division in the exam hall.

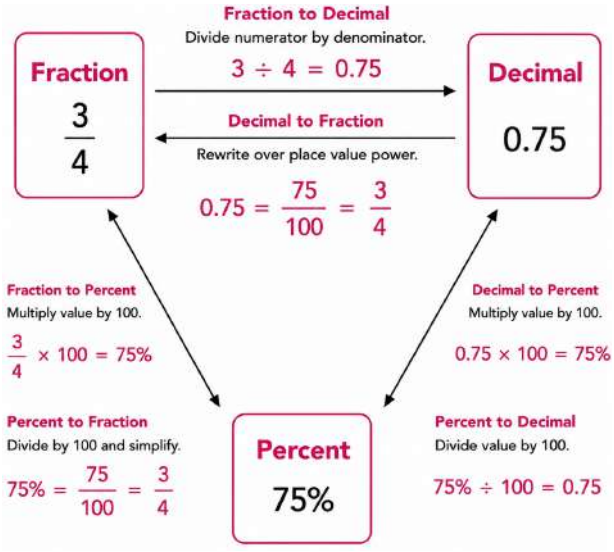


Figure 3.1 The three forms of the same number — fraction, decimal and per cent — and the conversions between them.

★ EXAM SHORTCUT

To find x % of N, compute (x · N) / 100. To find what per cent A is of B, compute (A / B) · 100. These two patterns alone cover the great majority of single-step Questions.

$$x\ \% = x / 100 = 0.0x \quad \text{(e.g. } 7\% = 0.07\text{)} \qquad (3.1)$$

3.2 Single Increase and Decrease

If a quantity Q is increased by p %, the new value is Q (1 + p/100). If it is decreased by p %, the new value is Q (1 − p/100). Always **multiply by the factor**, not by the per cent.

$$Q \text{ after a + p \%} = Q\,(1 + p / 100) \qquad (3.2)$$

$$Q \text{ after a − p \%} = Q\,(1 - p / 100) \qquad (3.3)$$

EXAMPLE 3.1

A bottle holds one litre of milk. A boy drinks 20 % of whatever is in the bottle in one sip. How much milk is left after the third sip?

(a) 0.40 L (b) 0.51 L
(c) 0.60 L (d) 0.80 L

Solution. Each sip removes 20 % and leaves 80 %, so after each sip the bottle keeps a factor of 0.80 of what it had. After three sips the remaining volume is 1 · 0.80 · 0.80 · 0.80 = 0.512 L ≈ 0.51 L. Answer (b).

3.3 Successive Percentage Changes — Why ± 20 Does Not Cancel

When two per cent changes act one after the other, the second one acts on the result of the first — not on the original. A 20 % rise followed by a 20 % cut therefore does not return to where we started.

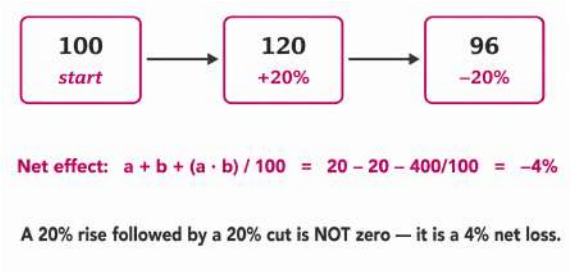


Figure 3.2 Two successive percentage changes — the second always operates on the value left after the first.

$$\text{net change} = a + b + \frac{a \cdot b}{100} \quad \text{(signs included)} \qquad (3.4)$$

SIGN CONVENTION

Take an increase as positive and a decrease as negative. So a 30 % rise followed by a 25 % fall gives net = 30 + (−25) + (30 · −25)/100 = 30 − 25 − 7.5 = −2.5 % — a 2.5 % net decrease.

EXAMPLE 3.2

A theatre raises the number of viewers by 30 % after cutting the ticket price by 25 % per person. What is the percentage change in total earnings?

(a) No change (b) 2.5 % decrease
(c) 2.5 % increase (d) 7.5 % increase

Solution. Earnings = (price) × (viewers). The price falls 25 %, the count rises 30 %. Using (3.4): net = 30 + (−25) + (30 · −25)/100 = −2.5 %. Total earnings fall by 2.5 %. Answer (b).

EXAMPLE 3.3

By selling two articles at the same price, a shopkeeper gains 20 % on one and loses 20 % on the other. What is the net outcome on the two transactions taken together?

(a) No gain, no loss (b) Net gain of 4 %
(c) Net loss of 4 % (d) Net loss of 20 %

Solution. Let the common selling price be 100. Cost of the gain-item is 100 / 1.20 = 83.33; cost of the loss-item is 100 / 0.80 = 125. Total cost = 208.33, total sale price = 200, loss = 8.33. As a percentage of cost: 8.33 / 208.33 ≈ 4 %. The result is general — selling equal-priced items at the same per cent gain and loss always yields a net loss of (p / 10)² per cent. Answer (c).

★ EXAM SHORTCUT

The 'same selling price, equal per cent gain & loss' trap appears year after year. The net is ALWAYS a loss, of size (p / 10)² per cent. So ±10 % → 1 % loss, ±20 % → 4 % loss, ±30 % → 9 % loss.

3.4 Reverse Percentage Problems

Many exam questions give the value after a per cent change and ask for the original. The trap is to subtract or add the same per cent — but the per cent was taken of the original, not of the new value.

$$\text{Original} = \text{Final} \div (1 \pm p / 100) \qquad (3.5)$$

EXAMPLE 3.4

A tank is so designed that '40 % empty' holds exactly 40 L more water than '40 % full'. What is the full capacity of the tank?

(a) 100 L (b) 120 L
(c) 150 L (d) 200 L

Solution. Let the capacity be C. '40 % empty' means 60 % full = 0.60 C; '40 % full' means 0.40 C. The difference is 0.20 C = 40 L, so C = 200 L. Answer (d).

3.5 Compound Growth and Decay

If a quantity changes by the same per cent r every period, after n periods it has been **multiplied n times** by the factor (1 + r/100).

$$Q_n = Q_0(1 + r/100)^n \quad (3.6)$$

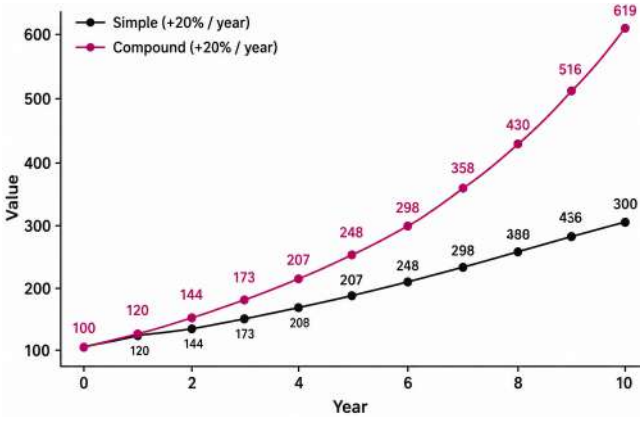


Figure 3.3 Compound growth (magenta) pulls clear of simple growth (black) once n exceeds three or four periods.

EXAMPLE 3.5

Two rabbit colonies start at the same size. Colony A grows at 25 % per year while colony B grows at 50 % per year. What is the ratio of B to A after two years?

(a) 25 : 4 (b) 9 : 4

(c) 9 : 5 (d) 36 : 25

Solution. After two years A is scaled by $1.25^2 = 1.5625$ and B by $1.50^2 = 2.25$. The ratio B : A is 2.25 : 1.5625 = 225 : 156.25 = 36 : 25. Answer (d).

DID YOU KNOW

If a quantity decays by p % every period, it never strictly reaches zero — it merely shrinks by the factor (1 − p/100) each step. This is the same geometric-series mathematics that describes radioactive decay, drug clearance and population decline.

3.6 Per Cent Errors and How They Multiply

If a physical quantity y depends on a measurement x through a power, $y = k \cdot x^n$, then a small per cent error in x produces an n-times bigger per cent error in y. This is the single most useful experimental shortcut for life-science and physical-science exam questions alike.

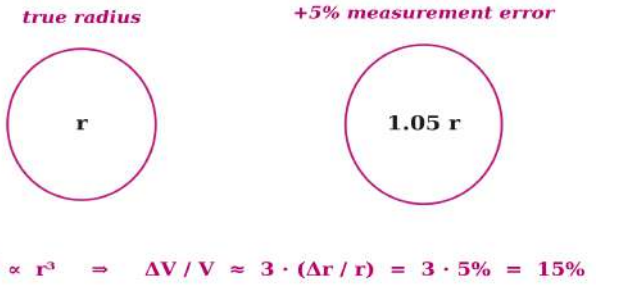


Figure 3.4 A 5 % error in measuring the radius of a sphere becomes a 15 % error in the volume because V varies as the cube of r.

$$\text{If } y = k \cdot x^n \text{ then } \Delta y / y \approx |n| \cdot \Delta x / x \quad (3.7)$$

EXAMPLE 3.6

The radius of a sphere is measured with a 5 % uncertainty. What is the resulting uncertainty in the volume calculated from this radius?

(a) 5 % (b) 10 %

(c) 15 % (d) 125 %

Solution. V = (4/3) π r^3, so the exponent is 3. Apply (3.7): ΔV / V ≈ 3 · 5 % = 15 %. Answer (c).

FOR A QUOTIENT Y = A / B

Per cent errors of independent measurements add in quadrature for a quotient. To leading order, the per cent error in y is approximately the sum of the per cent errors in a and b (or the square root of the sum of squares, when the errors are random and independent).

3.7 Replacing Part of a Mixture

A standard chemistry/biology setting: a container holds volume V of pure liquid; we remove a volume x and replace it with the same volume of water (or solvent). After n such removals and refills, the fraction of the original liquid that remains is

$$\text{Fraction remaining} = (1 - x / V)^n \quad (3.8)$$

EXAMPLE 3.7

A vendor has 50 L of pure milk. After each customer collects 5 L he tops the can up with 5 L of water. What per cent of milk does the fifth customer get?

(a) 59.0 % (b) 62.5 %

(c) 65.6 % (d) 72.0 %

Solution. Each cycle leaves a fraction (1 − 5/50) = 0.90 of the previous milk. After four complete cycles the milk fraction is $0.90^4 \approx 0.656$. The fifth customer therefore receives a solution that is ≈ 65.6 % milk. Answer (c).

★ EXAM SHORTCUT

Whenever the question reads 'removes x and refills with water n times', reach straight for (3.8). Many exam questions disguise this as 'alcohol diluted three times' or 'oil drawn off and replaced' — the recipe is identical.

Quick Recap

- Per cent is parts-per-hundred: $x\ \% = x / 100 = 0.0x$.
- Increase by p % → multiply by $(1 + p/100)$. Decrease → multiply by $(1 - p/100)$.
- Successive changes a, b: net $= a + b + (a \cdot b)/100$ with proper signs.
- Same SP, equal ± p % gain/loss ⇒ net loss $= (p / 10)^2$ per cent.
- Reverse %: Original = Final ÷ $(1 \pm p/100)$.
- Compound: $Q_n = Q_0 (1 + r/100)^n$.
- Error in derived quantity $y = k \cdot x^n \Rightarrow \Delta y / y \approx |n| \cdot \Delta x / x$.
- Mixture replacement n times: fraction remaining $= (1 - x/V)^n$.

Practice

Attempt all 10 questions within 15 minutes.

1. What is 7 / 8 expressed as a per cent?
   (a) 72.5 % (b) 75.0 %
   (c) 82.5 % (d) 87.5 %
2. If a salary of Rs 24 000 is increased by 12.5 %, the new salary is
   (a) Rs 26 500 (b) Rs 27 000
   (c) Rs 27 500 (d) Rs 27 800
3. A number is first increased by 25 % and then decreased by 25 %. The net change is
   (a) 0 % (b) 6.25 % decrease
   (c) 6.25 % increase (d) 12.5 % decrease
4. A book's price after a 20 % discount is Rs 480. What was the marked price?
   (a) Rs 540 (b) Rs 576
   (c) Rs 600 (d) Rs 620
5. Two articles are sold at the same price; one at a 10 % gain and the other at a 10 % loss. The overall result is
   (a) 1 % gain (b) 1 % loss
   (c) no change (d) 10 % loss
6. The diameter of a ball is measured with a 2 % error. The percentage error in its volume is approximately
   (a) 2 % (b) 4 %
   (c) 6 % (d) 8 %
7. A bacterial culture grows by 50 % every hour. By what factor does it grow in 3 hours?
   (a) 1.5 (b) 2.25
   (c) 3.0 (d) 3.375
8. A 100 mL flask contains a solute. 20 mL is removed and replaced with water; this is done three times. What fraction of the solute remains?
   (a) 0.512 (b) 0.640
   (c) 0.729 (d) 0.800
9. The numbers of seats for three subjects are in the ratio 2 : 3 : 4. If each is raised by 50 %, the new ratio is
   (a) 1 : 1 : 1 (b) 2 : 3 : 4
   (c) 4 : 3 : 2 (d) 3 : 2 : 4
10. 100 kg of cucumbers contain 99 % water by mass. After drying, the water content drops to 98 %. What is the new total mass?
    (a) 50 kg (b) 98 kg
    (c) 99 kg (d) 10 kg

Solutions

1. $7/8 = 0.875 = 87.5$ %. Answer (d).
2. $24\,000 \cdot 1.125 = 27\,000$. Answer (b).
3. Net $= 25 - 25 - (25 \cdot 25)/100 = -6.25$ %. Answer (b).
4. Marked $= 480 / 0.80 = 600$. Answer (c).
5. Standard result: net loss $= (10/10)^2 = 1$ %. Answer (b).
6. Volume $\propto d^3$, so $\Delta V/V \approx 3 \cdot 2\ \% = 6$ %. Answer (c).
7. $1.5^3 = 3.375$. Answer (d).
8. $(1 - 20/100)^3 = 0.8^3 = 0.512$. Answer (a).
9. Scaling every term by the same factor leaves the ratio unchanged. Answer (b).
10. The non-water (solid) mass is constant at 1 kg. If water is 98 %, then solid is 2 %. So 1 kg = 2 % of new mass ⇒ new mass = 50 kg. Answer (a).

14.4 The Ternary Plot — Three Components in One Triangle

When three quantities always sum to a fixed total (often 100 %), they have only two free parameters.

A **ternary plot** displays them inside an equilateral triangle: each vertex stands for '100 % of one component', and a point inside is read by following the gridlines parallel to the opposite side.

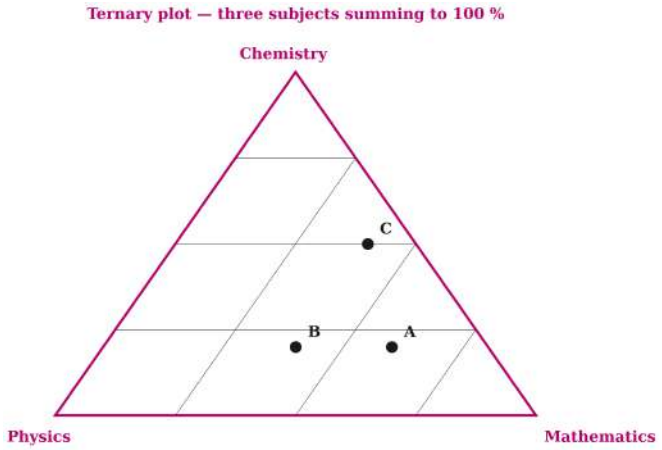


Figure 14.3 A ternary plot of three students' marks in Physics, Mathematics and Chemistry. Points C, B and A lie at different combinations summing to 100 marks.

- To read a point's value on one component, follow the gridline that runs PARALLEL to the opposite side until it meets the axis labelled with that component.
- The three component values for any point must sum to 100 % (or to the stated total).
- A vertex represents 100 % of that component and 0 % of the other two.

EXAMPLE 14.4

Refer to Figure 14.3 (a student's total is 100). Student C has 50 marks in Mathematics and 40 marks in Chemistry. How many marks did C score in Physics?

(a) 0 (b) 10

(c) 20 (d) 30

Solution. The three subject marks must sum to 100. So Physics = 100 − 50 − 40 = 10 marks. Answer (b).

14.5 Population Pyramids

A **population pyramid** is a back-to-back horizontal bar chart that plots age-bin counts (or percentages) for males on the left and females on the right. The pyramid's SHAPE reveals the demographic regime at a glance.

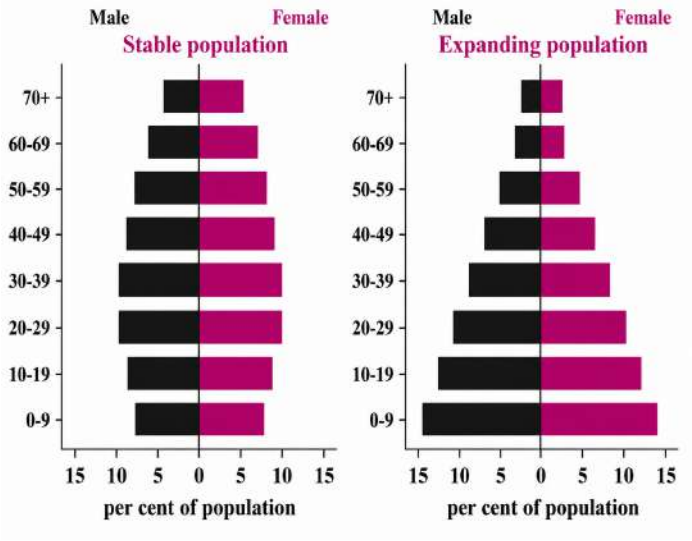


Figure 14.4 Two population pyramids. The left one — bars of roughly equal length across age bins — represents a stable population. The right one — wide base tapering rapidly upward — represents a rapidly expanding (young) population.

- Wide base, narrow top: high birth rate, young population, expanding.
- Roughly rectangular: similar numbers in every age bin — stable population.
- Narrow base, wider middle: low birth rate, ageing population, declining.
- Sex imbalance shows as left/right asymmetry within an age bin.

EXAMPLE 14.5

Refer to Figure 14.4. Which pyramid corresponds to the population with the higher child-dependency ratio (children per working-age adult)?

(a) The stable population on the left (b) The expanding population on the right

(c) Both have the same ratio (d) Cannot be determined from the figure

Solution. The expanding pyramid's base (0–9 and 10–19 bins) is much wider than the middle age-bins, so the ratio of children to working-age adults is markedly higher there than in the stable pyramid. Answer (b).

14.6 Box Plots — A Five-Number Summary

A box plot reduces a data set to five numbers: minimum, lower quartile Q_1, median, upper quartile Q_3, and maximum. The box covers Q_1 to Q_3 (the middle 50 % of the data), the whiskers reach down to the minimum and up to the maximum, and a magenta line inside the box marks the median. The box plot is the cleanest way to compare two or three distributions side by side.

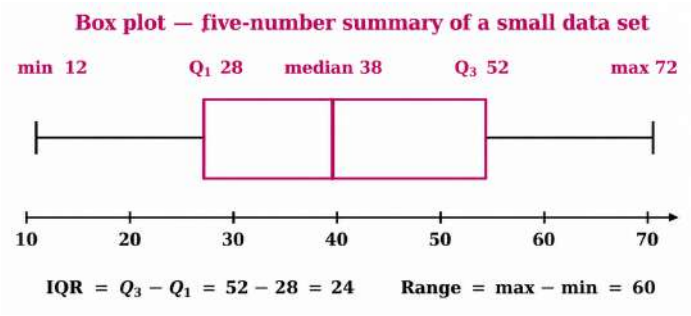


Figure 14.5 Box plot showing minimum 12, Q_1 28, median 38, Q_3 52 and maximum 72. The box width — the inter-quartile range (IQR) — is 24.

$$\text{Inter-quartile range (IQR)} = Q_3 - Q_1 \quad (14.3)$$

- A long whisker on one side signals skew toward that side.
- Points far beyond the whiskers (often beyond $Q_3 + 1.5 \times$ IQR or $Q_1 - 1.5 \times$ IQR) are flagged as outliers and shown as dots.
- When comparing two boxes: differences in median, differences in IQR width, and differences in whisker length each tell something distinct.

EXAMPLE 14.6

Refer to Figure 14.5. What fraction of the data lies inside the box?

(a) one-quarter (b) one-half
(c) three-quarters (d) all

Solution. The box spans Q_1 to Q_3 — by construction the middle 50 % of the data. So one-half of the observations lie inside the box. Answer (b).

14.7 Density and Compound Measures

Several exam question display two quantities side by side — area and population, mass and volume, energy and time — and ask for the implied ratio (density, concentration, power). The calculation is straightforward once you know which quantity is numerator and which is denominator.

$$\text{Population density} = \text{population} / \text{area} \quad (14.4)$$

COMPARING FOUR STATES

If you are given areas and populations of four states, the one with the highest population/area ratio is the most densely populated — not the one with the largest population, nor the smallest area.

DID YOU KNOW

The Nightingale rose chart — a precursor of the modern radar/polar bar chart, invented by Florence Nightingale in 1858 — used wedges of variable radius to show seasonal mortality among British soldiers in the Crimean War. Its dramatic visual contrast helped persuade Parliament to fund military-hospital reforms.

Quick Recap

- Venn diagram: each circle is a set; intersections are overlaps; outside is 'none'.
- Inclusion–exclusion (3 sets): $|A \cup B \cup C| = |A|+|B|+|C|$ − pairwise overlaps + triple overlap.
- Sankey: band widths are proportional to flow volumes; compare bands between two sinks or two sources.
- Ternary: three components sum to 100 %; read by following gridlines parallel to the opposite side.
- Population pyramid: wide base = young/expanding; rectangular = stable; narrow base = ageing/declining.
- Box plot: 5-number summary; box spans Q_1–Q_3 (middle 50 %); whiskers to min/max; median line inside.
- Density-style exam question: compute population / area (or analogous ratio); largest ratio wins.

Practice

Attempt all 13 questions within 15 minutes. All questions use Figures 14.1–14.5 from this chapter.

1. Refer to Figure 14.1. The total number of players is
   (a) 83 (b) 97
   (c) 103 (d) 110
2. Refer to Figure 14.1. The number of players who play exactly one sport is
   (a) 55 (b) 65
   (c) 75 (d) 85

3. Refer to Figure 14.1. The number of players who play ONLY Cricket is
(a) 20 (b) 25
(c) 30 (d) 44

4. Refer to Figure 14.1. The total number of cricket players (alone or with other sports) is
(a) 48 (b) 52
(c) 60 (d) 70

5. Refer to Figure 14.1. The number of players who play AT LEAST two sports is
(a) 20 (b) 24
(c) 28 (d) 32

6. Of 100 students who speak at least one of two languages, 60 speak English, 70 speak Hindi. The number who speak both is
(a) 20 (b) 30
(c) 40 (d) 50

7. In a class of 80 students, 50 study Maths, 40 study Physics and 30 study Chemistry. Twenty study Maths and Physics, 15 study Maths and Chemistry, 10 study Physics and Chemistry, and 5 study all three. The number who study at least one subject is
(a) 75 (b) 78
(c) 80 (d) 85

8. Refer to Figure 14.2. Across both genders, which grain is the most preferred?
(a) Rice (b) Wheat
(c) Millet (d) Cannot tell from the diagram

9. Refer to Figure 14.3. Student A's marks in (Physics, Mathematics, Chemistry) are closest to
(a) (20, 60, 20) (b) (20, 20, 60)
(c) (40, 40, 20) (d) (60, 20, 20)

10. In a ternary plot of three components summing to 100, a point at the centroid of the triangle represents the composition
(a) 33.3, 33.3, 33.3 (b) 50, 25, 25
(c) 0, 0, 100 (d) 100, 0, 0

11. Which of the following pyramid shapes most strongly suggests a stable population?
(a) Wide base tapering steeply to the top (b) Roughly rectangular across all age bins
(c) Narrow base widening towards the middle (d) Inverted — narrow at base, wide at the top

12. In a population pyramid, a noticeably longer male bar than female bar in a particular age bin indicates
(a) A higher overall birth rate (b) More males than females in that age bin
(c) Higher male mortality in that age bin (d) A typing error in the chart

13. In a box plot, the fraction of data points enclosed within the box (between Q_1 and Q_3) is approximately
(a) 25 % (b) 50 %
(c) 75 % (d) 100 %

Solutions

1. Sum of all seven region counts: 30 + 25 + 20 + 10 + 8 + 6 + 4 = 103. **Answer (c).**

2. Only-cricket + only-football + only-basketball = 30 + 25 + 20 = 75. **Answer (c).**

3. The non-overlap part of the Cricket circle is labelled 30. **Answer (c).**

4. Cricket = only-cricket + C∩F-only + C∩B-only + triple = 30 + 10 + 8 + 4 = 52. **Answer (b).**

5. Exactly two = 10 + 8 + 6 = 24. All three = 4. At least two = 24 + 4 = 28. **Answer (c).**

6. By (14.1): 100 = 60 + 70 − both, so both = 30. **Answer (b).**

7. By (14.2): 50 + 40 + 30 − 20 − 10 − 15 + 5 = 80 (just covering the class). **Answer (c).**

8. The widest sink (largest total inflow) is the most-preferred grain. Wheat receives more total band-width than Rice or Millet — Wheat is the most preferred overall. **Answer (b).**

9. Point A sits 20 % from the Physics axis, 60 % toward the Mathematics vertex and 20 % toward the Chemistry vertex. So (Phy, Maths, Chem) = (20, 60, 20). **Answer (a).**

10. The centroid is equidistant from all three vertices, so each component is 100 % / 3 ≈ 33.3 %. **Answer (a).**

11. Stable populations have similar counts at every age — a roughly rectangular pyramid. **Answer (b).**

12. Bar lengths are proportional to counts; a longer male bar in an age bin means there are simply more males than females in that bin. **Answer (b).**

13. By construction, Q_1 and Q_3 are the 25th and 75th percentiles; the box holds the middle 50 % of the data. **Answer (b).**

CHAPTER

15 Shapes, Lines and Angles

Angles · Parallel lines · Triangles · Pythagoras · Polygons · Quadrilaterals

IN THIS CHAPTER

Geometry questions on the paper test recognition far more than calculation. Once you spot a pair of parallel lines crossed by a transversal, an exterior angle of a triangle, a right-angled triangle whose sides fit a known Pythagorean triple, or a regular polygon whose angle sum follows the $(n - 2) \times 180°$ rule, the answer is one short step away. This chapter walks through every standard shape, the angle pairs that surround them, and the small handful of theorems that crack almost every plane-geometry question.

- *name and use the basic angle pairs — vertically opposite, supplementary, complementary;*
- *apply corresponding, alternate and co-interior angle rules to parallel-line problems;*
- *compute every interior or exterior angle of any triangle from the 180° / exterior-angle rules;*
- *recognise the 3–4–5, 5–12–13 and 8–15–17 Pythagorean triples and use them in seconds;*
- *compute the interior- and exterior-angle sums of any regular or irregular polygon.*

15.1 Points, Lines and Angles

A **point** has position but no size; a **line** is the shortest path through two distinct points and extends without limit in both directions; a **ray** starts at a point and extends in one direction; a **line segment** is the portion of a line between two points and so has a definite length. Two rays sharing a common endpoint form an **angle**; the common endpoint is the vertex and the rays are the arms.

- Acute angle — strictly between 0° and 90°.
- Right angle — exactly 90°; the two arms are perpendicular.
- Obtuse angle — between 90° and 180°.
- Straight angle — exactly 180°.
- Reflex angle — between 180° and 360°.

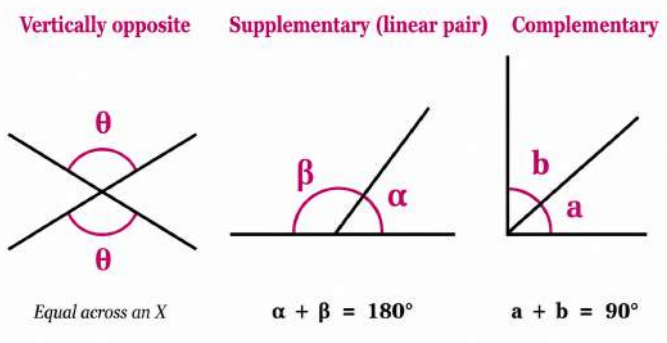


Figure 15.1 Three common angle pairs — vertically opposite angles are equal, supplementary angles sum to 180°, and complementary angles sum to 90°.

- ***Vertically opposite*** angles (across a crossing) are equal.
- ***Supplementary*** angles (on a straight line) sum to 180°.
- ***Complementary*** angles (on a right-angle arm) sum to 90°.
- ***Adjacent*** angles share an arm and a vertex but do not overlap.

EXAMPLE 15.1

Two lines intersect so that one of the four angles formed is 40°. The other three angles, in some order, are

(a) 40°, 50°, 140° (b) 40°, 140°, 140°

(c) 50°, 130°, 140° (d) 40°, 100°, 140°

Solution. When two lines cross, vertically opposite angles are equal and adjacent angles are supplementary. So the four angles are 40°, 140°, 40°, 140°. The three angles other than the given 40° are 140°, 40°, 140°. Answer (b).

15.2 Parallel Lines and Transversals

Two distinct lines in the same plane that never meet are **parallel**. A third line crossing both is a **transversal**, and it creates eight angles in two clusters of four. The angle pairs among them follow three handy rules.

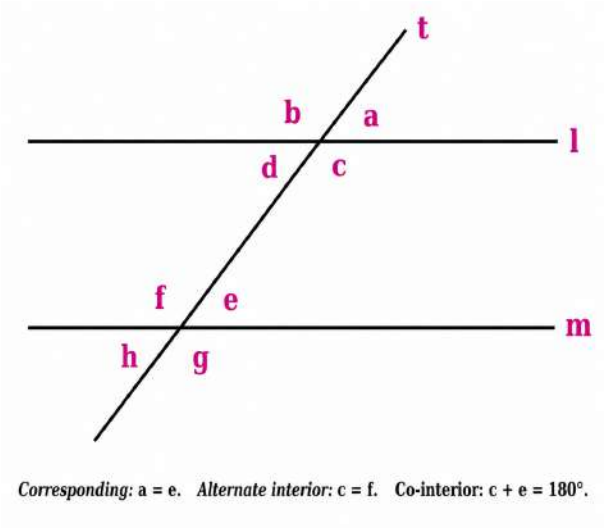


Figure 15.2 Two parallel lines L and m cut by a transversal t. Equal angle pairs are corresponding (a, e), alternate interior (c, e) and alternate exterior (a, g); co-interior angles (c, g) are supplementary.

- ***Corresponding*** angles — same side of the transversal, same side of their line — are equal.
- ***Alternate interior*** angles — between the two parallels, opposite sides of the transversal — are equal.
- ***Co-interior*** (same-side interior) angles sum to 180°.

EXAMPLE 15.2

In Figure 15.2, if angle a is 70°, what are angles e and g?

(a) e = 70°, g = 70° (b) e = 70°, g = 110°

(c) e = 110°, g = 70° (d) e = 110°, g = 110°

Solution. a and e are corresponding angles, so e = a = 70°. The angle g sits on the same straight line as e and is therefore supplementary: g = 180° − 70° = 110°. Answer (b).

★ EXAM SHORTCUT

'Co-interior angles supplementary' is the easiest sufficient condition to use when you are asked to PROVE two lines are parallel. If you can show two same-side interior angles add up to 180°, the lines must be parallel.

15.3 Triangles — Properties and Types

A triangle has three sides and three interior angles that always **sum to 180°**. An **exterior angle** of a triangle equals the sum of the two interior angles not adjacent to it.

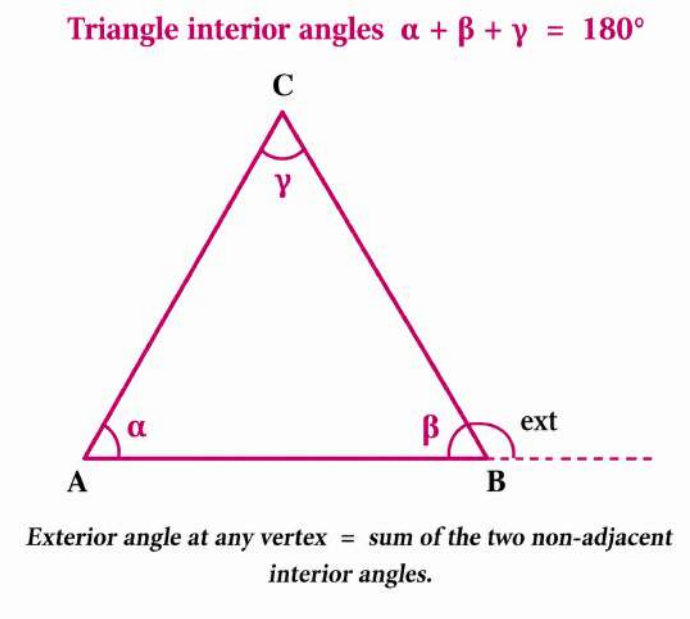


Figure 15.3 Interior angles of any triangle add up to 180°. The exterior angle at any vertex equals the sum of the two non-adjacent interior angles.

- By sides — equilateral (all sides equal), isosceles (two sides equal), scalene (no two sides equal).
- By angles — acute (all angles less than 90°), right (one angle exactly 90°), obtuse (one angle more than 90°).
- ***Triangle inequality*** — the sum of any two sides is strictly greater than the third.
- The longest side of a triangle is opposite the largest angle.

EXAMPLE 15.3

In triangle ABC, angle A is larger than angle C and smaller than angle B by the same amount. If angle B is 67°, what is angle C?

(a) 45° (b) 47°

(c) 53° (d) 60°

Solution. Let the common amount be d. Then angle A = angle C + d and angle B = angle A + d. So angle B − angle C = 2d. Setting B = 67° gives angle C = 67° − 2d. Also A + B + C = 180°, so (C + d) + (C + 2d) + C = 180° ⇒ 3 C + 3 d = 180° ⇒ C + d = 60°, i.e. A = 60°. Then d = 67° − 60° = 7°, and C = A − d = 60° − 7° = 53°. Answer (b).

15.4 The Pythagorean Theorem and the Common Triples

For a right-angled triangle with legs a and b and hypotenuse c, **Pythagoras**'s theorem says **$a^2 + b^2 = c^2$**. Three integer triples appear so often in exam question that they are worth memorising.

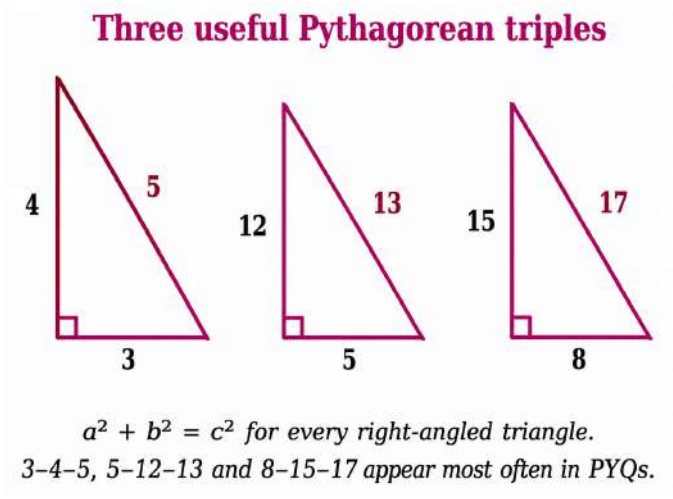


Figure 15.4 The three Pythagorean triples that recur in the paper — 3–4–5, 5–12–13 and 8–15–17. Each can be scaled by any positive integer to give a new triple.

$$a^2 + b^2 = c^2 \quad (15.1)$$

SCALING A TRIPLE

Any positive multiple of a triple is also a triple: 3–4–5 scales to 6–8–10, 9–12–15, 12–16–20 …; 5–12–13 to 10–24–26, 15–36–39; 8–15–17 to 16–30–34, 24–45–51.

EXAMPLE 15.4

A right-angled triangle has integer sides with hypotenuse 17 cm. What is its area (in cm²)?

(a) 30 (b) 60

(c) 84 (d) 120

Solution. The 8–15–17 triple gives legs of 8 cm and 15 cm. Area = (1/2) × 8 × 15 = 60. Answer (b).

15.5 Similar and Congruent Triangles

Two triangles are **similar** if their corresponding angles are equal and their corresponding sides are in the same ratio. They are **congruent** if their corresponding sides are EQUAL — one is essentially a rigid copy of the other.

- Similarity tests: ***AA*** (two pairs of equal angles), ***SAS*** (two pairs of proportional sides with the included angle equal), ***SSS*** (all three pairs of sides proportional).
- Congruence tests: SSS, SAS, ASA, AAS, RHS (right angle, hypotenuse, side — for right triangles).

If △ABC ~ △PQR, then $AB / PQ = BC / QR = CA / RP$ (15.2)

Areas of similar triangles = $(\text{ratio of sides})^2$ (15.3)

EXAMPLE 15.5

Two similar triangles have sides in the ratio 3 : 5. The ratio of their areas is

(a) 3 : 5 (b) 5 : 3

(c) 9 : 25 (d) 25 : 9

Solution. By (15.3) area ratio = $(3 / 5)^2 = 9 / 25$. Answer (c).

15.6 Polygons — Interior and Exterior Angles

A polygon with n sides has its interior angles summing to **(n − 2) × 180°**. The sum of its exterior angles is always **360°**, independent of n. For a regular polygon every interior angle equals (n − 2) × 180° / n, and every exterior angle equals 360° / n.

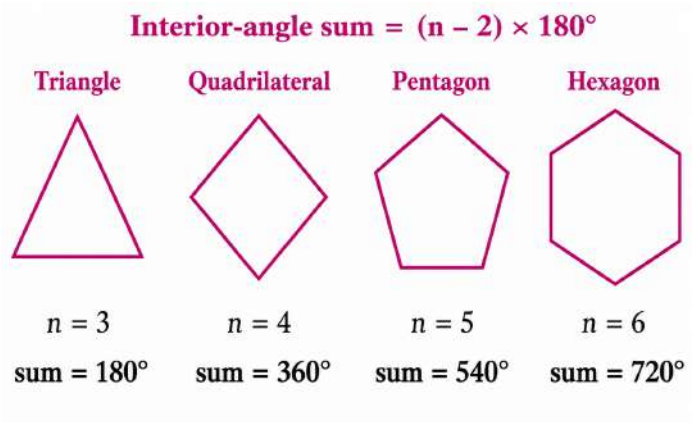


Figure 15.5 Interior-angle sums for the four most common regular polygons. Each extra side adds 180° to the total.

Interior-angle sum = (n − 2) × 180° (15.4)

One interior angle of a regular n-gon

$$= (n - 2) \times 180° / n \quad (15.5)$$

One exterior angle of a regular n-gon

$$= 360° / n \quad (15.6)$$

EXAMPLE 15.6

The sum of the interior angles of a regular octagon is

(a) 360° (b) 900°

(c) 1 080° (d) 1 260°

Solution. An octagon has n = 8, so by (15.4) the interior-angle sum is (8 − 2) × 180° = 6 × 180° = 1 080°. Answer (c).

EXAMPLE 15.7

Each exterior angle of a regular polygon is 24°. The polygon has how many sides?

(a) 12 (b) 15

(c) 18 (d) 24

Solution. By (15.6), 360° / n = 24° ⇒ n = 15. Answer (b).

15.7 Quadrilaterals

A quadrilateral is a four-sided polygon. Its interior angles sum to (4 − 2) × 180° = **360°**. The common species are arranged in a hierarchy from most-symmetric (square) to least.

- Square — four equal sides, four right angles. Diagonals are equal and bisect each other at right angles.
- Rectangle — opposite sides equal, four right angles. Diagonals are equal and bisect each other.
- Rhombus — four equal sides, opposite angles equal. Diagonals are unequal, bisect each other at right angles.
- Parallelogram — opposite sides equal and parallel; opposite angles equal; diagonals bisect each other (but are not generally equal).
- Trapezium — exactly one pair of opposite sides parallel.
- Kite — two pairs of adjacent equal sides; one diagonal bisects the other perpendicularly.

HIERARCHY

Square ⊂ rectangle ⊂ parallelogram. Square ⊂ rhombus ⊂ parallelogram. Every property of a parallelogram is inherited by rectangles, rhombuses and squares.

EXAMPLE 15.8

The two adjacent angles of a parallelogram differ by 40°. The smaller angle is

(a) 50° (b) 60°

(c) 70° (d) 80°

Solution. Adjacent angles of a parallelogram are supplementary, summing to 180°. Let them be x and x + 40°. Then 2 x + 40° = 180° ⇒ 2 x = 140° ⇒ x = 70°. Answer (c).

15.8 Angles of Elevation and Depression

Looking up at a target makes an angle of **elevation** with the horizontal; looking down makes an angle of **depression**. These are alternate interior angles between the horizontal line at the observer's eye and a parallel horizontal line at the target, so the two angles are equal — useful in problems where you swap viewpoints.

$$\text{In a right triangle: } \tan(\text{angle}) = (\text{opposite leg}) / (\text{adjacent leg}) \quad (15.7)$$

EXAMPLE 15.9

From a point on the ground, the angle of elevation to the top of a 30 m tall tower is 45°. How far from the foot of the tower is the observation point?

(a) 15 m (b) 20 m

(c) 30 m (d) 60 m

Solution. At 45° the tan equals 1, so opposite leg = adjacent leg. Height = horizontal distance = 30 m. Answer (c).

DID YOU KNOW

The architect of the Great Pyramid at Giza used the 3–4–5 triangle (knotted on a 12-loop rope) to lay out perfect right angles, several thousand years before Pythagoras would give the theorem its name.

Quick Recap

- Vertically opposite angles equal; supplementary on a line (sum 180°); complementary for a right angle (sum 90°).
- Parallel lines + transversal: corresponding equal, alternate interior equal, co-interior supplementary.
- Triangle interior angles sum to 180°; exterior angle = sum of two opposite interior.
- Triangle inequality: any two sides sum to more than the third.
- Pythagoras $a^2 + b^2 = c^2$. Memorise 3–4–5, 5–12–13, 8–15–17 and their multiples.
- Similar triangles: side ratio constant; area ratio = (side ratio)^2.
- n-gon interior-angle sum = $(n - 2) \times 180°$; regular n-gon exterior angle = $360° / n$.
- Square ⊂ rectangle ⊂ parallelogram and square ⊂ rhombus ⊂ parallelogram.
- Angle of elevation at the observer equals angle of depression at the target — they are alternate interior angles.

Practice

Attempt all 15 questions within 15 minutes. Mark only one option per question. Solutions follow.

1. Two angles are complementary. One of them is 35°. The other is
   (a) 35° (b) 55°
   (c) 65° (d) 145°

2. Two angles are supplementary. One of them is 110°. The other is
   (a) 20° (b) 30°
   (c) 70° (d) 80°

3. Two straight lines intersect, forming four angles. If one of the angles is 60°, the sum of the other three angles is
   (a) 240° (b) 300°
   (c) 240° or 300° (d) 180°

4. In a triangle the three angles are in the ratio 2 : 3 : 4. The largest angle is
   (a) 60° (b) 75°
   (c) 80° (d) 90°

5. An exterior angle of a triangle is 120°. Of the two non-adjacent interior angles, one is 50°. The other is
   (a) 50° (b) 60°
   (c) 70° (d) 130°

6. A right-angled triangle has legs of length 5 cm and 12 cm. Its hypotenuse measures
   (a) 7 cm (b) 13 cm
   (c) 17 cm (d) 60 cm

7. A right-angled triangle has legs 9 cm and 12 cm. Its hypotenuse is
   (a) 13 cm (b) 15 cm
   (c) 16 cm (d) 18 cm

8. Which of the following sets of three numbers CANNOT be the lengths of the sides of a triangle?
   (a) 3, 4, 5 (b) 5, 6, 8
   (c) 2, 3, 6 (d) 7, 8, 9

9. Two similar triangles have sides in ratio 2 : 3. The ratio of their areas is
   (a) 2 : 3 (b) 4 : 9
   (c) 6 : 9 (d) 8 : 27

10. The sum of interior angles of a hexagon is
    (a) 360° (b) 540°
    (c) 720° (d) 1 080°

11. Each interior angle of a regular pentagon is
    (a) 72° (b) 90°
    (c) 108° (d) 120°

12. If each interior angle of a regular polygon is 144°, the polygon has how many sides?
    (a) 8 (b) 9
    (c) 10 (d) 12

13. In a parallelogram, one angle is 75°. The other three angles are
    (a) 75°, 75°, 75° (b) 75°, 105°, 105°
    (c) 85°, 95°, 75° (d) 75°, 75°, 105°

14. From a point 50 m from the foot of a tower, the angle of elevation of the top of the tower is 30°. The height of the tower is
    (a) 25 m (b) $25/\sqrt{3}$ m
    (c) $50/\sqrt{3}$ m (d) 100 m

15. The diagonal of a square is 8 cm. The area of the square (in cm^2) is
    (a) 8 (b) 16
    (c) 32 (d) 64

Solutions

1. Complementary angles sum to 90°: 90° − 35° = 55°. Answer (b).
2. Supplementary sums to 180°: 180° − 110° = 70°. Answer (c).
3. The four angles are 60°, 120°, 60°, 120°. The other three sum to 60° + 120° + 120° = 300°. Answer (b).
4. Sum 2k + 3k + 4k = 9k = 180° ⇒ k = 20°. Largest = 4k = 80°. Answer (c).
5. Exterior angle = sum of two non-adjacent interior angles: 120° = 50° + x ⇒ x = 70°. Answer (c).
6. 5–12–13 triple. Hypotenuse = 13. Answer (b).
7. 9–12–15 is a multiple of 3–4–5 (×3). Hypotenuse = 15. Answer (b).
8. For a triangle, the sum of any two sides must exceed the third. 2 + 3 = 5 < 6, so these three cannot form a triangle. Answer (c).
9. Areas of similar triangles scale as the square of the side ratio: $(2 / 3)^2 = 4 / 9$. Answer (b).
10. n = 6 ⇒ (6 − 2) × 180° = 720°. Answer (c).
11. Sum = (5 − 2) × 180° = 540°. Each = 540° / 5 = 108°. Answer (c).
12. Exterior angle = 180° − 144° = 36°. n = 360° / 36° = 10. Answer (c).
13. Opposite angles in a parallelogram are equal, adjacent angles are supplementary. So the four angles are 75°, 105°, 75°, 105°. Answer (b).
14. tan 30° = h / 50 ⇒ h = 50 × tan 30° = 50 / √3 m. Answer (c).
15. For a square of side a, diagonal = a√2; so a = 8/√2 = 4√2, and area = $a^2 = 32$. Answer (c).

CHAPTER

16 The Circle Toolbox

Anatomy · Circumference · Tangents · Chord theorems · Inscribed angles

IN THIS CHAPTER

The circle has its own short catalogue of theorems — each useful in dozens of disguises. Once you know the anatomy of a circle, the perpendicular-from-centre rule for chords, the tangent-radius right angle, the inscribed-angle theorem and the power of a point, almost every single-circle exam questions collapses to one short step. This chapter handles the toolbox; Chapter 17 will turn it loose on two-circle problems, coin rotations and mixed figures.

- *name every part of a circle and compute its circumference, area, arc length and sector area;*
- *apply the perpendicular-from-the-centre and equal-chords-equidistant theorems;*
- *use the tangent-radius right angle and the two-tangent equal-length rule;*
- *apply the inscribed-angle theorem and the cyclic-quadrilateral 180° rule;*
- *use the power-of-a-point relation for intersecting chords and tangent-secant configurations.*

16.1 Anatomy of a Circle

A **circle** is the set of points in a plane at a fixed distance — the radius — from a fixed point called the **centre**. Every other named feature of a circle is built from this single definition.

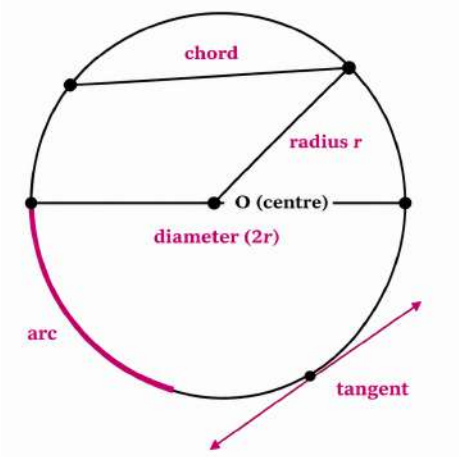


Figure 16.1 The named parts of a circle — centre, radius, diameter, chord, arc and tangent.

- Centre O — the fixed point.
- Radius r — any segment from the centre to a point on the circle.
- Diameter — any chord that passes through the centre; its length is 2 r.
- Chord — a segment joining any two points on the circle.
- Arc — a portion of the circle (a curve segment) between two points.
- Tangent — a line that touches the circle at exactly one point.
- Sector — the region bounded by two radii and the arc between them.
- Segment — the region bounded by a chord and one of the arcs it cuts off.

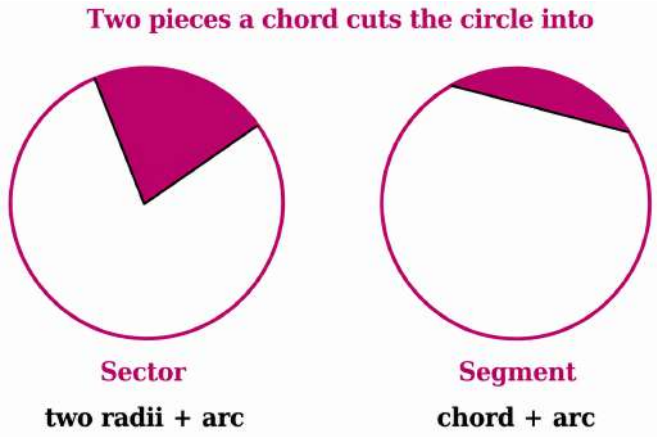


Figure 16.2 Sector versus segment — a sector is bounded by two radii and an arc; a segment is bounded by a chord and an arc.

16.2 Circumference, Area, Arc Length and Sector Area

The two basic measurements of a full circle are its **circumference** (perimeter) and its **area**. An arc and a sector are easy: just scale by the fraction $\theta / 360°$ of the full angle.

$$\text{Circumference} = 2\pi r \qquad \text{Area} = \pi r^2 \quad (16.1)$$

$$\text{Arc length (central angle } \theta°) = (\theta / 360°) \times 2\pi r \quad (16.2)$$

$$\text{Sector area (central angle } \theta°) = (\theta / 360°) \times \pi r^2 \quad (16.3)$$

TWO USEFUL CONVERSIONS

$\theta° = \pi\,\theta / 180°$ rad. In radians, arc length $= r\,\theta$ and sector area $= (1/2)\,r^2\,\theta$ — the neatest pair of formulas in elementary trigonometry. Worth knowing both forms.

EXAMPLE 16.1

A sector of a circle has radius 10 cm and central angle 72°. Its area (in cm²) is

(a) 10π (b) 12π

(c) 20π (d) 25π

Solution. By (16.3) area $= (72 / 360) \times \pi \times 100 = (1 / 5) \times 100\,\pi = 20\,\pi$ cm². Answer (c).

16.3 Tangents — Lines That Touch

A **tangent** meets the circle at exactly one point — the point of contact. Three theorems exhaust everything an aptitude paper will ask:

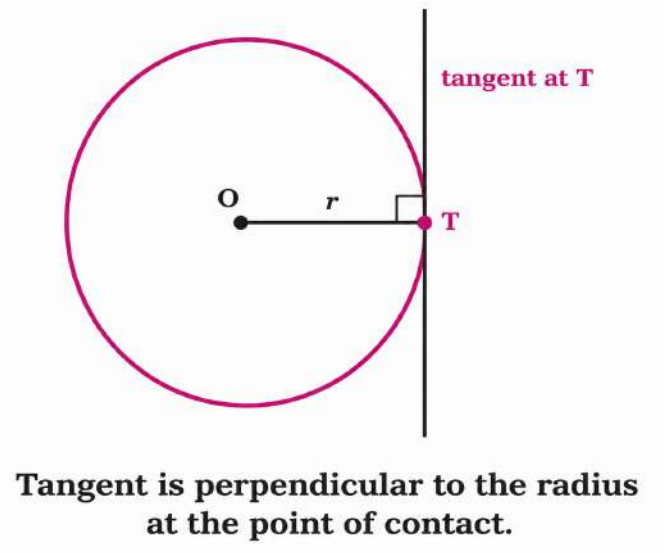


Figure 16.3 The tangent at any point of a circle is perpendicular to the radius drawn to that point.

- ***Tangent ⊥ radius*** at the point of contact.
- From any external point, ***two tangents*** to a circle can be drawn, and their lengths are equal.
- Tangent-secant from an external point P: $PT^2 = PA \cdot PB$, where PT is the tangent length and A, B are the two intersection points of the secant with the circle.Tangent-secant from an external point P: $PT^2 = PA \cdot PB$, where PT is the tangent length and A, B are the two intersection points of the secant with the circle.

EXAMPLE 16.2

From an external point P, the tangent to a circle has length 8 cm. A secant from the same point cuts the circle at A and B with PA = 4 cm. What is the full secant length PB?

(a) 12 cm (b) 14 cm

(c) 16 cm (d) 20 cm

Solution. By the tangent-secant rule, $PT^2 = PA \times PB$. So $64 = 4 \times PB \Rightarrow PB = 16$ cm. Answer (c).

16.4 Chord Theorems

Two facts about chords come up again and again — the perpendicular-bisector rule and the equal-chords-equidistant rule.