PATTERN OF EXAMINATION

Particulars	Details
Mode of examination	Computer Based Test (CBT)
Duration	3 hours
Number of papers	30 test papers
Sections	General Aptitude (GA) + Candidate's Selected Subject(s)
Types of questions	(a) Multiple Choice Questions (MCQ) (b) Multiple Select Questions (MSQ) (c) Numerical Answer Type (NAT)
Test of abilities	(a) Recall (b) Comprehension (c) Application (d) Analysis and Synthesis
Number of questions	10 (General Aptitude) +55 (Subject) = 65 Questions
Distribution of marks in all papers except AR, CY, DA, EY, GG, MA, PH, ST, XH and XL	General Aptitude: 15 marks Engineering Mathematics": 13 marks Subject Questions: 72 marks Total: 100 marks ("XE includes an Engineering Mathematics section XE-A of 15 marks)
Distribution of marks in papers AR, CY,DA, EY, GG, MA, PH, ST, XH and XL	General Aptitude: 15 marks Subject Questions: 85 marks Total: 100 marks
Marking scheme	Questions carry either 1 mark or 2 marks
Negative marking	For a wrong answer to an MCQ, there will be negative marking. For a 1-mark MCQ, 1/3 will be deducted for a wrong answer. Likewise, for 2-mark MCQ, 2/3 will be deducted for a wrong answer. There is no negative marking for wrong answers to MSQ and NAT questions. There is no partial marking for any question.

*PwD candidates with benchmark disability more than 40% are eligible for compensatory time of one hour. PwD candidates with less than 40% disability and having difficulty in writing are eligible for compensatory time subject to production of a certificate as stated in Appendix-I of the guidelines issued by the Ministry of Social Justice and Empowerment.

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CALCULUS

CHAPTER 1

Unit 1.1: Function of Several Variables

MCQ (Multiple Choice Questions)

[GATE 2006]

1. Let $f: R^2 \to R$ be defined by

$$f(x,y) = \begin{cases} \frac{x^2y}{x^4+y^2} & for \quad (x,y) \neq (0,0) \\ 0 & for \quad (x,y) = (0,0) \end{cases}$$

Then, the directional derivative of f at $(0,0)$ in the direction of the vector $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is

(a) $\frac{1}{\sqrt{2}}$ (b) $\frac{1}{2}$

(c) $\frac{1}{2\sqrt{2}}$ (d) $\frac{1}{4\sqrt{2}}$

[GATE 2006]

2. Consider the hemisphere $x^2 + y^2 + (z-2)^2 = 9,\ 2 \leq x \leq 5$ and the vector field $\vec{F}(x,y,z) = x\vec{\imath} + y\vec{\jmath} + (z-2)\vec{k}$. The surface integral $\iint \left(\vec{F} \cdot \vec{n}\right) d\sigma$, evaluated over the hemisphere with denoting the unit outward normal is

(a) 9π (b) 27π

(c) 54π (d) 162π

[GATE 2007]

3. Define $f: \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1, & if\ xy = 0 \\ 2, & otherwise. \end{cases}$$

If $S = \{(x,y): f \text{ is continuous at the point (x, y)}\}$, then

(a) S is open (b) S is connected

(c) $S = \phi$ (d) S is closed

Statement for Linked Answer Questions 4 & 5:

Suppose $E = \{(x,y): xy \neq 0\}$. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} 0, & if\ xy = 0 \\ y\sin\left(\frac{1}{x}\right) + x\sin\left(\frac{1}{y}\right), & otherwise, \end{cases}$$

Let S_1 be the set of points in $\mathbb{R}^2$ where f_x exists and S_2 be the set of points in $\mathbb{R}^2$ where f_y exists. Also, let E_1 be the set of points where f_x is continuous and E_2 be the set of points where f_y is continuous.

[GATE 2007]

4. S_1 and S_2 are given by

(a) $S_1 = E \cup \{(x,y): y = 0\}, S_2 = E \cup \{(x,y): x = 0\}$

(b) $S_1 = E \cup \{(x,y): x = 0\}, S_2 = E \cup \{(x,y): y = 0\}$

(c) $S_1 = S_2 = \mathbb{R}^2$

(d) $S_1 = S_2 = E \cup \{(0,0)\}$

[GATE 2007]

5. E_1 and E_2 are given by

(a) $E_1 = E_2 = S_1 \cap S_2$

(b) $E_1 = E_2 = S_1 \cap S_2 \backslash \{(0,0)\}$

(c) $E_1 = S_1, E_2 = S_2$

(d) $E_1 = S_2, E_2 = S_1$

[GATE 2008]

6. For $(x,y) \in R^2$, let

$$f(x,y) = \begin{cases} \frac{2xy}{x^2+y^2} if\ (x,y) \neq (0,0), \\ 0\ if\ (x,y) = (0,0). \end{cases}$$

Then

(a) f_x and f_y exist at (0,0), and f is continuous at (0,0)

(b) f_x and f_y exist at (0,0), and f is discontinuous at (0,0)

(c) f_x and f_y do not exist at (0,0), and f is continuous at (0,0)

(d) f_x and f_y do not exist at (0,0) and f is discontinuous (0,0)

Statement for Linked Answer Question 7 and 8:

Let $f(x,y) = kxy - x^3y - xy^3$ for $(x,y) \in \mathbb{R}^2$, where k is a real constant. The directional derivative of f at the point (1, 2) in the direction of the unit vector $u = \left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ is $\frac{15}{\sqrt{2}}$.

[GATE 2009]

7. The value of k is

(a) 2 (b) 4

(c) 1 (d) -2

[GATE 2009]

8. The value of f at a local minimum in the rectangular region $R = \left\{(x,y) \in \mathbb{R}^2 : |x| < \frac{3}{2}, |y| < \frac{3}{2}\right\}$ is

(a) -2 (b) -3

(c) $\frac{-7}{8}$ (d) 0

[GATE 2017]

9. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2+y^2}, & x \neq 0, \\ 0, & x = 0 \end{cases}$$

Then, at $(0,0)$,

(a) f is continuous and the directional derivative of f does NOT exist in some direction
(b) f is NOT continuous and the directional derivatives of f exist in all directions
(c) f is NOT differentiable and the directional derivatives of f exist in all directions
(d) f is differentiable

[GATE 2019]

10. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = x^6 - 2x^2y - x^4y + 2y^2$
(R is the set of all real numbers and $\mathbb{R}^2 = \{(x,y): x,y \in \mathbb{R}\}$
Which one of the following statements is TRUE?
(a) f has a local maximum at origin
(b) f has a local minimum at origin
(c) f has a saddle point at origin
(d) The origin is not a critical point of f

[GATE 2019]

11. Let $g: \mathbb{R}^1 \to \mathbb{R}^2$ be a function defined by $g(x,y) = (e^x \cos y, e^x \sin y)$ and $(a,b) = g\left(1, \frac{\pi}{3}\right)$,
(R is the set of all real numbers and $\mathbb{R}^2 = \{(x,y): x,y \in \mathbb{R}\}$ which one of the following statements is TRUE?
(a) g is injective
(b) If h is the continuous inverse of g, defined in some neighbourhood of $(a,b) \in \mathbb{R}^2$, such that $h(a,b) = \left(1, \frac{\pi}{3}\right)$, then the Jacobian of h at (a,b) is e^2
(c) If h is the continuous inverse of g, defined in some neighbourhood of $(a,b) \in \mathbb{R}^2$, such that $h(a,b) = 1$, then the Jacobian of h at (a,b) is e^{-2}
(d) g is surjective

[GATE 2019]

12. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\frac{1}{x^2+y^2}, & if\ (x,y) \neq (0,0) \\ 0, & if\ (x,y) = (0,0) \end{cases}$$

Consider the following statements:
I. The partial derivatives exist at, $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$, but are unbounded in any neighbourhood of (0,0)
II. f is continuous but not differentiable at $(0,0)$.
III. f is not continuous at $(0,0)$.
IV. f is differentiable at (0,0).
($\mathbb{R}$ is the set of all real numbers and $\mathbb{R}^2 = \{(x,y): x,y \in \mathbb{R}\}$
Which of the above statements is/are TRUE?
(a) I and II only (b) I and IV only
(c) IV only (d) III only

[GATE 2020]

13. If $(4,0)$ and $\left(0, -\frac{1}{2}\right)$ are critical points of the function $f(x,y) = 5 - (\alpha+\beta)x^2 + \beta y^2 + (\alpha+1)y^3 + x^3$, where $\alpha, \beta \in \mathbb{R}$, then
(a) $\left(4, -\frac{1}{2}\right)$ is a point of local maxima of f
(b) $\left(4, -\frac{1}{2}\right)$ is a saddle point of f
(c) $\alpha = 4, \beta = 2$
(d) $\left(4, -\frac{1}{2}\right)$ is a point of local minima of f

[GATE 2021]

14. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be differentiable. Let $D_u f(0,0)\ and\ D_v f(0,0)$ Be the directional derivatives of f at (0,0) in the directions of the unit vectors $u = \left(\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}\right)$ And $v = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$, respectively. If $D_u f(0,0) = \sqrt{5}\ and\ d_v f(0,0) = \sqrt{2}$, then $\frac{af}{ax}(0,0) + \frac{af}{af}(0,0)$

[GATE 2021]

15. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y,) = \begin{cases} \sqrt{x^2+y^2}\ \sin\left(\frac{y^2}{x}\right) & if\ x \neq 0 \\ 0 & if\ x \neq 0 \end{cases}$
Consider the following statements:
P: f is continuous at (0,0) but f is not differentiable at (0,0)
Q: The directional derivative $D_u f(0,0) of\ f\ at\ (0,0)$ exists in the direction of every unit vector $u \in \mathbb{R}^2$
Then
(a) P is TRUE and Q is FALSE
(b) both P and Q are TRUE
(c) both P and Q are FALSE
(d) P is FALSE and Q is TRUE

[GATE 2021]

16. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be given by $f(x,y) = 4xy - 2x^2 - y^4$. Then f has
(a) two saddle points
(b) a point of local maximum and a point of local minimum
(c) a point of local maximum and a saddle point
(d) a point of local minimum and a saddle point

[GATE 2022]

17. If the function $f(x,y) = x^2 + xy + y^2 + \frac{1}{x} + \frac{1}{y}, x \neq 0, y \neq 0$ attains its local minimum value at the point (a,b), then the value of $a^3 + b^3$ is _____ (round off to TWO decimal places).

[GATE 2023]

18. Let $f, g: \mathbb{R}^2 \to \mathbb{R}$ be defined by
$f(x,y) = x^2 - 2xy^2\ and\ g(x,y) = 4x^4 - 5x^2y + y^2$
for all $(x,y) \in \mathbb{R}^2$.

Consider the following statements:

P: f has a saddle point at $(0,0)$.

Q: g has a saddle point at $(0,0)$.

Then

(a) both P and Q are TRUE

(b) P is FALSE but Q is TRUE

(c) P is TRUE but Q is FALSE

(d) both P and Q are FALSE

[GATE 2023]

19. Let $f: \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \frac{2x^2y}{x^2+y^2} & (x,y) \neq (0,0) \\ 0 & (x,y) = (0,0) \end{cases}$$

Then

(a) the directional derivative of f at $(0,0)$ in the direction of $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ is $\frac{1}{\sqrt{2}}$

(b) the directional derivative of f at $(0,0)$ in the direction of $(0,1)$ is 1

(c) the directional derivative of f at (0,0) in the direction of $(1,0)$ is 0

(d) f is NOT differentiable at $(0,0)$

[GATE 2018]

20. Let $u(x,y,z) = x^2 - 2y + 4z^2$ for $(x,y,z) \in \mathbb{R}^3$. Then the directional derivative of u in the direction $\frac{3}{5}\hat{\imath} - \frac{4}{5}\hat{k}$ at the point (5, 1, 0) is _____.

[GATE 2019]

21. The temperature $T : \mathbb{R}^3 \backslash \{(0,0,0)\} \to \mathbb{R}$ at any point $P(x,y,z)$ is inversely proportional to the square of the distance of P from the origin. If the value of the temperature T at the point $R(0,0,1)$ is $\sqrt{3}$, then the rate of change of T at the point $Q(1,1,2)$ in the direction of $\overrightarrow{QR}$ is equal to _____ decimal).

(round off to 2 places of

($\mathbb{R}$ is the set of all real numbers, $\mathbb{R}^3 = \{(x,y,z): x,y,z \in \mathbb{R}\}$ *and* $R^3 \backslash \{(0,0,0)\}$ *denote* R^3 excluding the origin)

[GATE 2022]

22. The rate of change of $f(x,y,z) = x + x\cos z - y\sin z + y$ at P_0 in the direction from $P_0(2,-1,0)$ to $P_1(0,1,2)$ is_____

[GATE 2012]

23. The maximum value of the function $f(x,y,z) = xyz$ subject to the constraint $xy + yz + zx - a = 0, a > 0$ is

(a) $a^{\frac{3}{2}}$ (b) $\left(\frac{a}{3}\right)^{3/2}$

(c) $\left(\frac{3}{a}\right)^{3/2}$ (d) $\left(\frac{3a}{2}\right)^{3/2}$

[GATE 2015]

24. Let $D = \{(x,y) \in \mathbb{R}^2 : 1 \leq x \leq 1000, 1 \leq y \leq 1000\}$. Define

$f(x,y) = \frac{xy}{2} + \frac{500}{x} + \frac{500}{y}$.

Then the minimum value of f on D is equal to _____

[GATE 2022]

25. The maximum value of $f(x,y) = 49 - x^2 - y^2$ on the line $x + 3y = 10$ is _____

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
(a)	(c)	(a)	(a)	(b)	(b)	(b)	(a)	(c)	(c)
11	**12**	**13**	**14**	**15**	**16**	**17**	**18**	**19**	**20**
(a), (c)	(b)	(b)	4	(a)	(a)	0.67	(a)	(a),(c), (d)	6
21	**22**	**23**	**24**	**25**					
0.22	0	(b)	150	39					

EXPLANATIONS

1. **Explanation:**

Let $c = (0,0), u = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$

$f'(c,u) = \lim_{h\to 0} \frac{f(c+hu)-f(c)}{h} = \lim_{h\to 0} \frac{f\left(\frac{h}{\sqrt{2}}, \frac{h}{\sqrt{2}}\right) - f(0,0)}{h}$

$\lim_{h\to 0} \frac{\left(\frac{\frac{h^2}{2}\left(\frac{h}{\sqrt{2}}\right)}{\frac{h^4}{4}+\frac{h^2}{2}}\right) - 0}{h} = \lim_{h\to 0} \left(\frac{\frac{h^3}{2\sqrt{2}}}{\frac{h^4+2h^2}{4}}\right)\frac{1}{h}$

$= \lim_{h\to 0} \frac{h^2}{2\sqrt{2}}\left(\frac{4}{h^4+2h^2}\right) = \lim_{h\to 0} \frac{1}{\sqrt{2}}\left(\frac{2}{h^2+2}\right) = \frac{1}{\sqrt{2}}$

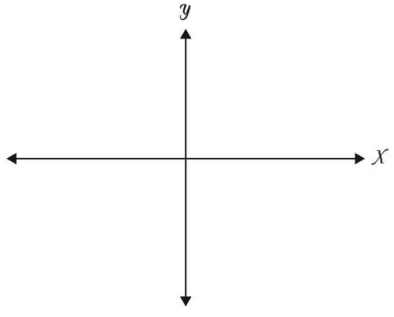
2. **Explanation:**

We know $\iint \left(\vec{F}.\vec{n}\right) d\sigma = \iiint_c divF\, dxdydz$. (where C is the region of hemisphere)

$\Rightarrow \iint \left(\vec{F}.\vec{n}\right) d\sigma = \iiint_c 3dxdydz = 3\iiint dxdydz = 3 \times \frac{2}{3}\pi \times (3)^3 = 54\pi$

($\because$ volume of hemisphere with radius $r = \frac{2}{3}\pi r^3$)

3. **Explanation:**



Here f is continuous at the point $xy \neq 0$

$S = \mathbb{R}^2 \backslash (\{(x,y): y = 0\} \cup \{(x,y): x = 0\})$ because $xy = 0 \Leftrightarrow x = 0$ or $y = 0$

Since, $\{(x,y): x = 0\} \cup \{(x,y): y = 0\}$ is closed

$\therefore$ S is open [$\because$ A set S is open iff S is closed]

4. **Explanation:**

$E = \{(x,y): xy \neq 0\}$

$f: \mathbb{R} \to \mathbb{R}$, defined as

$$f(x,y) = \begin{cases} 0, & if\ xy = 0 \\ y\sin\frac{1}{x} + x\sin\frac{1}{y}, & otherwise \end{cases}$$

$S_1 = \{(x,y) \in \mathbb{R}^2 | f_x\ exist\}, S_2 = \{(x,y) \in \mathbb{R}^2 | f_y\ exists\}$ on E,

$f_x = (-y/x^2)\cos\frac{1}{x} + \sin\frac{1}{y}, f_y = \sin\frac{1}{x} - \frac{x}{y^2}\cos\frac{1}{y}$

$\therefore$ on E, f_x, f_y both exist

Consider, $\{(x,y): y = 0\}$

$f_x(x,0) = \lim_{x \to x_1} \frac{f(x,0) - f(x_1,0)}{x - x_1} = 0$

f_x exist on $\{(x,y): y = 0\}$

Consider, $\{(x,y): x = 0\}$

$f_x(0,y) = \lim_{x \to 0} \frac{f(x,y) - f(0,y)}{x} = \lim_{x \to 0} \frac{y\sin\frac{1}{x} + x\sin\frac{1}{y} - 0}{x} =$

$\lim_{x \to 0} \frac{y\sin\frac{1}{x}}{x} + \lim_{x \to 0} \sin\frac{1}{y}$

But $\lim_{x \to 0} \frac{\sin\frac{1}{x}}{x}$ does not exist

$\therefore f_x$ does not exist on $\{(x,y): x = 0, y \neq 0\}$

$\Rightarrow S_1 = E \cup \{(x,y): y = 0\}$

Similarly, $S_2 = E \cup \{(x,y): x = 0\}$

5. **Explanation:**

$$f_x(x,y) = \begin{cases} 0, & \{(x,y): y = 0\} \\ \frac{-y}{x^2}\cos\frac{1}{x} + \sin\frac{1}{y} & on\ E \end{cases}$$

$$f_y(x,y) = \begin{cases} 0, & \{(x,y): x = 0\} \\ \frac{-y}{y^2}\cos\frac{1}{y} + \sin\frac{1}{x} & on\ E \end{cases}$$

Clearly f_x and f_y are continuous on E

Continuity of f_x at $(x,0)$

$f_x(x,0) = 0$

$\lim_{(x,y) \to (x,0)} f_x(x,y) = \lim_{(x,y) \to (x,0)} \left\{-\frac{y}{x^2}\cos\frac{1}{x} + \sin\frac{1}{y}\right\}$

$\lim_{(x,y) \to (x,0)} \underbrace{\frac{-y}{x^2}}_{\downarrow 0}\cos\frac{1}{x} + \lim_{(x,y) \to (x,0)} \underbrace{\sin\frac{1}{y}}_{\downarrow 0}$

$\therefore f_x$ is not continuous at $(x,0)$

$\therefore f_x$ is continuous only on E and $\{(0,0)\} \Rightarrow E_1 = S_1 \cap S_2 - \{(0,0)\}$

Similarly, we can check f_y is continuous only on E and $\{(0,0)\} \Rightarrow E_2 = S_1 \cap S_2 - \{(0,0)\}$

$\therefore E_1 = E_2 = S_1 \cap S_2 - \{(0,0)\}$

6. **Explanation:**

$f_x(0,0) = \lim_{x \to 0} \frac{f(x,0) - (0,0)}{x - 0} = 0$

$f_y(0,0) = \lim_{x \to 0} \frac{f(0,y) - f(0,0)}{y - 0} = 0$

Consider, $\lim_{\substack{x \to 0 \\ y \to 0}} f(x,y) = \lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{x^2 + y^2} = \lim_{x \to 0} \frac{2x.mx}{x^2 + m^2x^2} = \frac{2m}{1 + m^2}$

(for $y = mx$)

Which depends on m

$\therefore$ limit doesn't exist at $(0,0)$ and hence 'f' is discontinuous at $(0,0)$

7. **Explanation:**

$f(x,y) = kxy - x^3y - xy^3$

Directional derivative of f at the point (1.2) in the direction of $u\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)$ is given by

$\lim_{t \to 0} \frac{f(x+tu) - f(x)}{t - 0} \Rightarrow \lim_{t \to 0} \frac{f\left((1,2) + t\left(\frac{-1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}\right)\right) - f(1,2)}{t} = \frac{15}{\sqrt{2}}$

$\Rightarrow \lim_{t \to 0} \frac{f\left(1 - \frac{t}{\sqrt{2}}, 2 - \frac{t}{\sqrt{2}}\right) - f(1,2)}{t} = \frac{15}{\sqrt{2}}$

Where $f\left(1 - \frac{t}{\sqrt{2}}, 2 - \frac{t}{\sqrt{2}}\right) = k\left(1 - \frac{t}{\sqrt{2}}\right)\left(2 - \frac{t}{\sqrt{2}}\right) - \left(1 - \frac{t}{\sqrt{2}}\right)^3\left(2 - \frac{t}{\sqrt{2}}\right) - \left(1 - \frac{t}{\sqrt{2}}\right)\left(2 - \frac{t}{\sqrt{2}}\right)^3$ and $f(1,2) = (2k - 2 - 2^3)$

$\Rightarrow \lim_{t \to 0} \frac{\left(1 - \frac{t}{\sqrt{2}}\right)\left(2 - \frac{t}{\sqrt{2}}\right)\left(k\left(1 - \frac{t}{\sqrt{2}}\right)^2\left(2 - \frac{t}{\sqrt{2}}\right)^2\right) - (2k - 10)}{t} = \frac{15}{\sqrt{2}}$

$\Rightarrow \lim_{t \to 0} \frac{\left(2 - \frac{3t}{\sqrt{2}} + \frac{t^2}{2}\right)\left(k - \left(1 - \frac{t}{\sqrt{2}}\right)^2 - \left(2 - \frac{t}{\sqrt{2}}\right)^2\right) - (2k - 10)}{t} = \frac{15}{\sqrt{2}}$

Using L'Hospital rule and solving, we get

$\Rightarrow 2\left(0 - 2\left(\frac{-1}{\sqrt{2}}\right) - 2.2\left(\frac{-1}{\sqrt{2}}\right)\right) + \frac{-3}{\sqrt{2}}(k - 1 - 4) = \frac{15}{\sqrt{2}}$

$\Rightarrow 2\left(\frac{6}{\sqrt{2}}\right) - \frac{3}{\sqrt{2}}(k - 5) = \frac{15}{\sqrt{2}} \Rightarrow \frac{12}{\sqrt{2}} - \frac{3}{\sqrt{2}}k + \frac{15}{\sqrt{2}} = \frac{15}{\sqrt{2}}$

$\Rightarrow k = 4$

8. **Explanation:**

Using result, $f(x,y)$ attains a local minima at (a,b) if $f_x(a,b) = f_y(a,b) = 0$ and

$f_{xx}(a,b)f_{yy}(a,b) - \left(f_{xy}(a,b)\right)^2 > 0$ and $f_{xx}(a,b)$ or $f_{yy}(a,b) > 0$

As $f(x,y) = 4xy - x^3y - xy^3$

$\therefore f_x = 4y - 3x^2y - y^3 = 0 \Rightarrow y = 0$ or $3x^2 + y^2 = 4$ and $f_y = 4x - x^3 - 3xy^2 = 0$

$\Rightarrow x = 0$ or $x^2 = 3y^2 = 4 \Rightarrow$ f may attain its local minima at $(0,0), (\pm 1, \pm 1)$

$f_{xx} = -6xy, f_{yy} = -6xy, f_{xy} = 4 - 3x^2 - 3y^2$

$\therefore f_{xx}f_{yy} - f_{xy}^2 = 36x^2y^2 - (4 - 3x^2 - 3y^2)^2$

$= 36 - (4 - 3 - 3)^2$ at $(\pm 1, \pm 1)$

$= 36 - 4 = 32 > 0$ and $f_{xx}f_{yy} - f_{xy}^2 = -4 < 0$ at $(0,0)$

$f_{xx}(-1,1) = 6 > 0$ and $f_{xx}(1,-1) = 6 > 0$

$\therefore$ f attain its local minima at $(-1,1)$ and $(1,-1)$

$\therefore f(1,-1) = -4 + 1 + 1 = -2$ and $f(-1,1) = -4 + 1 + 1 = -2$

9. **Explanation:**

$f: \mathbb{R}^2 \to \mathbb{R}$

$$f(x,y) = \begin{cases} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2+y^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$\lim_{(x,y)\to(0,0)} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2+y^2}$

Here $\sin\left(\frac{y^2}{x}\right)$ is bounded

$\sqrt{x^2+y^2} \to 0$ as $(x,y) \to (0,0)$

$\Rightarrow \lim_{(x,y)\to(0,0)} \sin\left(\frac{y^2}{x}\right)\sqrt{x^2+y^2} = 0 = f(0)$

$f(x,y)$ is continuous at $(0,0)$

$D_u(F(\alpha)) = \lim_{n\to 0} \frac{F(\alpha+hu)-F(\alpha)}{n}$

$D_u(F(\alpha)) = \lim_{n\to 0} \frac{F(a+hu,b+ku_2)-F(a,b)}{h}$

$D_u(F(\alpha)) = \lim_{n\to 0} \frac{F(hu_1,hu_2)-F(0,0)}{n}$

$D_u(F(\alpha)) = \lim_{h\to 0} \frac{\sin\left(\frac{h^2u_2^2}{hu_1}\right)\sqrt{h^2u_1^2+h^2u_2^2}}{n}$

$D_u(F(\alpha)) = \lim_{h\to 0} \frac{\sin\left(\frac{u_2^2}{u_1}\right)h\sqrt{u_1^2+u_2^2}}{n}$

$D_u(F(\alpha)) = \lim_{h\to 0} \sin\left(\frac{hu_2^2}{u_1}\right)\sqrt{u_1^2+u_2^2}$

$D_uF(0,0) = 0$

Hence all directional derivative exists

$f_y(0,0) = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$

$= \lim_{k\to 0} \frac{\sin\left(\frac{k}{0}\right)\sqrt{0+k^2}}{k}$

DNE

$\Rightarrow f(x,y)$ is not differentiable

10. **Explanation:**

Let

$f(x,y) = x^6 - 2x^2y - x^4y + 2x^2$

Consider Hessian matrix

$$H = \begin{vmatrix} F_{xx} & F_{xy} \\ F_{yx} & F_{yy} \end{vmatrix}$$

$$H = \begin{vmatrix} 30x^4 - 4y - 12x^2y & -4x - 4x^3 \\ -4x - 4x^3 & 4 \end{vmatrix}$$

At $(0,0)$

$$H = \begin{vmatrix} 0 & 0 \\ 0 & 4 \end{vmatrix}$$

$|H| = 0$

Test fails

$F(x,cx^2) = (1-c)(x^4)(x^2-2c)$

For $0 < c < 1$, $F(x,cx^2) < 0$

$F(x,0) = x^6 > 0$

$\Rightarrow (0,0)$ is a saddle point

11. **Explanation:**

$g(x,y) = (e^x \cos y, e^x \sin y)$

$(a,b) = g\left(1,\pi/3\right) = \left(\frac{e}{2},\frac{\sqrt{3}e}{2}\right)$

$g(x,y) = (e^x \cos y, e^x \sin y)$

g is not injective

$g(x,0) = g(x,2\pi)$

g is not surjective since $(0,0)$ has no preimage

$$J(g(x)) = \begin{vmatrix} f_x & f_y \\ f_{2x} & f_{2y} \end{vmatrix} = \begin{vmatrix} e^x\cos y & -e^x \sin y \\ e^x \sin y & e^x \cos y \end{vmatrix}$$

$J(g(x)) = e^{2x}$

$J\left(g(x)\right) = e^{2x}$

$J\left(g\left(1,\pi/3\right)\right) = e^2$

N is continuous inverse of h Jacobean of (a,b) us equal to inverse of the Jocobian of g at $\left(1,\pi/3\right)$ is $\frac{1}{e^2}$

12. **Explanation:**

$$f(x,y) = \begin{cases} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right) & (x,y) \neq 0 \\ 0 & (x,y) = (0) \end{cases}$$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{n\to 0} \frac{f(h,0)-f(0,0)}{n}$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{n\to 0} \frac{h^2 \sin\left(\frac{1}{h^2}\right)-0}{h}$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \lim_{h\to 0} h\sin\left(\frac{1}{h^2}\right)$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = 0$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{f(0,k)-f(0,0)}{k}$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} \frac{k^2 \sin\left(\frac{1}{k^2}\right)}{k}$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0} k\sin\left(\frac{1}{k^2}\right)$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = 0$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} (x^2+y^2)\sin\left(\frac{1}{x^2+y^2}\right)$

Since $\sin\left(\frac{1}{x^2+y^2}\right)$ bounded

& $x^2+y^2 \to 0$ as $(x,y) = (0,0)$

$\Rightarrow \lim_{(x,y)\to(0,0)} f(x,y) = f(0,0)$

$\Rightarrow f(x,y)$ continuous

$f(a+h,b+k) - f(a,b) = Ah + Bk + h\phi_1(h,k) + k\phi_2(h,k)$

$f(h,k) - f(0,0) = Ah + Bk + h\phi_1(h,k) + k\phi_2(h,k)$

$\sqrt{h^2+k^2}\sin\left(\frac{1}{h^2+k^2}\right) = h\phi_1(h,k) + k\phi_2(h,k)$

$\phi_1(h,k) = \sqrt{1+\left(\frac{k}{h}\right)^2}\sin\left(\frac{1}{h^2+k^2}\right)$

$\lim_{(h,b)\to(0,0)} \phi_1(h,k) = \lim_{(h,k)\to(0,0)} \sqrt{1+\left(\frac{k}{h}\right)^2}\sin\left(\frac{1}{h^2+k^2}\right) =$
0

⇒ f is differentiable

Option (B) correct

13. **Explanation:**

$(4,0)\ and\ \left(0,-\frac{1}{2}\right)\ are\ critical\ points\ of$

$f(x,y) = 5-(\alpha+\beta)x^2+\beta y^2+(\alpha+1)y^3+x^3$

$f_x(x,y) = -2(\alpha+\beta)x+3x^2$

$f_y(x,y) = 2\beta y+3(\alpha+1)y^2$

$f_x(4,0) = 0\ and\ f_y(4,0) = 0$

$-8(\alpha+\beta)+48 = 0$

$\alpha+\beta = 6 \dots\dots\dots\dots (1)$

$f_x\left(0,-\frac{1}{2}\right) = 0\ and\ f_y\left(0,-\frac{1}{2}\right) = 0$

$3\alpha-4\beta = -3 \dots\dots\dots (2)$

$From\ (1)\ and\ (2)$

We get

$\alpha = \beta = 3$

$f(x,y) = 5-6x^2+3y^2+4y^3+x^3$

$f_{xx} = -12+6x \qquad f_{yy} = 6+24y$

$f_{xx}\left(4,-\frac{1}{2}\right) = 12 > 0$

$f_{xx}\left(4,-\frac{1}{2}\right) = -6 < 0$

$f_{xy}\left(4,-\frac{1}{2}\right) = 0$

$(f_{xx})(f_{yy})-(f_{xy})^2 = -72 < 0$

$\left(4,-\frac{1}{2}\right)$ is saddle point

14. **Explanation:**

$f: \mathbb{R}^2 \to \mathbb{R}$ differetialbe

$u = \left(\frac{1}{\sqrt{5}},\frac{2}{\sqrt{5}}\right),\ v = \left(\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$\nabla f.u\left(\frac{\partial f}{\partial x}i+\frac{\partial f}{\partial y}\right)\left(\frac{1}{\sqrt{5}}i+\frac{2}{\sqrt{5}}j\right)$

$\nabla f\ .u\ = \frac{1}{\sqrt{5}}\frac{\partial f}{\partial x}+\frac{2}{\sqrt{5}}\frac{\partial f}{\partial y}$

$\sqrt{5} = \frac{1}{\sqrt{5}}\frac{\partial f}{\partial x}+\frac{2}{\sqrt{5}}\frac{\partial f}{\partial y}$

$5 = \frac{\partial f}{\partial x}+2\frac{\partial f}{\partial y}$ (1)

$\nabla f.v = \left(\frac{\partial f}{\partial x}\hat{\imath}+\frac{\partial f}{\partial y}\hat{\jmath}\right)\left(\frac{1}{\sqrt{2}}\hat{\imath}-\frac{1}{\sqrt{2}}\hat{\jmath}\right)$

$\sqrt{2} = \frac{1}{\sqrt{2}}\frac{\partial f}{\partial x}-\frac{1}{\sqrt{2}}\frac{\partial f}{\partial y}$

$2 = \frac{\partial f}{\partial x}-\frac{\partial f}{\partial y}$ (2)

Subtract (2) form (1)

$3 = 3\frac{\partial f}{\partial y}$

$\frac{\partial f}{\partial y} = 1$

Form (2)

$5 = \frac{\partial f}{\partial x}+2(1)$

$\frac{\partial f}{\partial x} = 3$

$\frac{\partial f}{\partial x}+\frac{\partial f}{\partial y} = 4$

15. **Explanation:**

$f(x,y) = \begin{cases}\sqrt{x^2+y^2}\sin\left(\frac{y^2}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$

$\lim_{(x,y)\to(0,0)} f(x,y) = \lim_{(x,y)\to(0,0)} \sqrt{x^2+y^2}\sin\left(\frac{y^2}{x}\right)$

$\lim_{(x,y)\to(0,0)} f(x,y) = 0 = f(0)$

⇒ $f(x,y)$ is continuous

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{f(0,k)-f(0,0)}{k}$

$\left.\frac{\partial f}{\partial y}\right|_{(0,0)} = \lim_{k\to 0}\frac{k\sin\left(\frac{k^2}{2}\right)}{k}$

DNE

⇒ f is not differentiable

$D_u f(0,0) = \lim_{h\to 0}\frac{f(hu_1,hu_2)-f(0,0)}{h}$

$D_u f(0,0) = \lim_{h\to 0}\frac{\sqrt{h^2u_1^2+h^2u_2^2}\sin\left(\frac{h^2u_3^2}{hu_1}\right)}{h}$

$D_u f(0,0) = \lim_{h\to 0}\sqrt{u_1^2+u_2^2}\sin\left(\frac{hu_2^2}{u_1}\right)$

$D_u f(0,0) = 0$

⇒ directional derivative exists

16. **Explanation:**

$f(x,y) = 4xy-2x^2-y^4$

$f_x = 4y-4x \qquad f_y = 4x-4y^3$

$f_x = 0 \qquad f_y = 0$

$y = x \qquad x = y^3$

$x = y^3$

$x = x^3$

$x-x^3 = 0$

$x(x^2-1) = 0$

$x = 0$ or $x = \pm 1$

Critical points are $(0,0),(1,1),(-1,-1)$

$f_{xx} = -4, f_{xy} = 4, f_{yy} = -12y^2$

$f_{xx}f_{yy}-f_{xy}^2 = 0-(4)^2 = -16 < 0$

⇒ $(0,0)$ is saddle point

For $(1,1)$

$f_{xx} < 0$

But $f_{xx}f_{yy}-f_{xy}^2 = -4(-12)-16 = 32 > 0$