P R E F A C E

This **"M.sc Physics Entrance" Book Volume 1** - has been carefully designed to provide a comprehensive overview of topics from **Mathematical Methods, Solid State Physics and Electronics** for undergraduate and graduate students in the field of Physical Science.

The aim of this book is to provide a clear and concise understanding of the subject matter, without overwhelming the reader with unnecessary technical jargon. The language used throughout the book is simple and easy to understand, making it accessible to students with varying levels of knowledge and experience.

In addition to the text, the book includes a range of tables and diagrams that illustrate key concepts for better understanding. One of the key features of this book is its relevance to various competitive exams, such as IIT JAM, CUET (PG), JEST, TIFR, (Physical Science) exams. The book has been designed keeping in mind the syllabus and requirements of these exams, making it an ideal resource for students who are preparing for these tests.

We hope that this book will serve as a valuable resource for students studying bioscience and will help them to deepen their understanding of this fascinating subject. If you come across any misprints or errors, please do inform us, as we are committed to providing accurate and up-to-date information to our readers.

A C K N O W L E D G E M E N T

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SECTION – I

MATHEMATICAL METHODS

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MATHEMATICAL METHODS

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Chapter 1
Partial Derivative & Jacobian

Partial Differentiation and the Chain Rule

In this section we review and discuss certain notations and relations involving partial derivatives.

The more general case can be illustrated by considering a function $f(x, y, z)$ of three variables x, y and z. If y and z are held constant and only x is allowed to vary, the partial derivative of f with respect to x is denoted by $\frac{\partial f}{\partial x}$ and defined by

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x, y, z)}{\Delta x} \quad (1)$$

Similarly we define

$$\frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x, y, z)}{\Delta y} \quad (2)$$

$$\frac{\partial f}{\partial z} = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x, y, z)}{\Delta z} \quad (3)$$

These formulae are direct generalisations of the well known definition of the derivative of afunction $f(x)$ of one variable x

$$\frac{df}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} \quad (4)$$

Example

Let $f(x, y, z) = x^2yz + ye^z$, then

$$\frac{\partial f}{\partial x} = 2xyz$$

$$\frac{\partial f}{\partial y} = x^2z + e^z$$

$$\frac{\partial f}{\partial z} = x^2y + ye^z$$

The Chain Rule

We present the chain for the function $f(x, y, z)$. We consider 3 cases.

(1) If x, y and z are all functions of a single variable t, then f can be considered as a functionof t and

$$\frac{df}{dx} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt} \quad (5)$$

Example

Let $f(x, y, z) = x^2yz$, $x = e^t$, $y = t$ and $z = 1 + t$

Method 1: Substitute the expressions for x, y and z into f. This yields

$$f = e^{2t}t(1 + t) \quad (6)$$

and differentiate (6). This gives

$$\frac{df}{dt} = e^{2t}(2t + 1) + 2e^{2t}(1 + t) \quad (7)$$

Method 2 Use the chain rule (5). This gives

$\frac{df}{dt} = (2xyz)e^t + x^2z + x^2y$ (8)

Substituting the expressions for x, y and z into (8) gives (7).

(2) More generally if x, y and z are all functions of two (or more variables), say$s$ and t, thenwe can consider f as a function of s and $t$and

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$ (9)

In writing (9) we consider the variables$t$ and s as 'associated' in the sense that in$\frac{\partial f}{\partial t}$ we assumethat s is held constant. Similarly the variables$x$, y and$z$ are associated in the sense that in$\frac{\partial f}{\partial x}$weassume that y and$z$ are held constant. Alternatively (and this is more elegant and safer) we candefine

$F(s,t) = f[x(s,t), y(s,t), z(s,t)]$ (10)

and write

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t}$ (11)

Example Let$f(x,y,z) = xyz$, $x = st$, $y = s + t$, $z = t$. We consider f as a function of s and t and wewant to calculate$\frac{\partial f}{\partial t}$.

Method 1: Substitute the expressions for x, y and z into f. This gives

$F(s,t) = s^2t^2 + st^3$ (12)

Differentiating 12 with respect to t gives

$\frac{dF}{dt} = 2s^2t + 3st^2$ (13)

Method 2: Use the chain rule (9) or (10). Here we use (9). This gives

$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial t} = yzs + xz + xy$ (14)

Substituting the expressions for x, y and z into (14) gives (13).

(3) We know suppose that y and z are functions of x. Then $f(x,y,z)$ can be considered as afunction of x and we have

$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx}$ (15)

Example

Let $f(x,y,z) = xyz, y = x$ and $z = x^2$

Method 1: Substitute the expressions for y and $z$into f. This gives

$f = xyz = x^4$ (16)

and

$\frac{df}{dx} = 4x^3$ (17)

Method 2: Use the chain rule 15. This gives

$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{dy}{dx} + \frac{\partial f}{\partial z}\frac{dz}{dx} = yz + xz + 2x^2y$ (18)

Substituting the expressions for y and z into (18) gives (17).

Higher Order Derivatives

Consider the function $f(x, y, z)$. We define the second order partial derivatives by the formula

$$\frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\frac{\partial f}{\partial x} \tag{19}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\frac{\partial f}{\partial y} \tag{20}$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\frac{\partial f}{\partial x} \tag{21}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\frac{\partial f}{\partial y} \tag{22}$$

It is not always true that

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x} \tag{23}$$

However (23) holds if all the partial derivatives of f up to second order are continuous. This condition is usually satisfied in applications and in particular in all the examples considered in this course.

The following alternative notation for partial derivatives is often convenient and more economical.

$$f_x = \frac{\partial f}{\partial x}$$

$$f_y = \frac{\partial f}{\partial y}$$

$$f_{xx} = \frac{\partial^2 f}{\partial x^2}$$

$$f_{yy} = \frac{\partial^2 f}{\partial y^2}$$

$$f_{xy} = \frac{\partial^2 f}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial^2 f}{\partial x \partial y}$$

Example

Let $f(x, y) = x^3 \cos y$. Then

$$\frac{\partial f}{\partial x} = 3x^2 \cos y$$

$$\frac{\partial f}{\partial y} = -x^3 \sin y$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \cos y$$

$$\frac{\partial^2 f}{\partial y^2} = -x^3 \cos y$$

$$\frac{\partial^2 f}{\partial y \partial x} = -3x^2 \sin y$$

$$\frac{\partial^2 f}{\partial x \partial y} = -3x^2 \sin y$$

We note that $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$ in accordance with our previous remark.

Jacobians

Changing variable is something we come across very often in Integration. There are many reasons for changing variables but the main reason for changing variables is to convert the integrand into something simpler and also to transform the region into another region which is easy to work with. When we convert into a new set of variables it is not always easy to find the limits. So, before we move into changing variables with multiple integrals we first need to see how the region may change with a change of variables. In order to change variables in an integration we will need the **Jacobian** of the transformation.

If $f_1, f_2, \ldots, f_n$ are n differentiable functions of n variables $x_1, x_2, \ldots, x_n$ then the determinant

$$\begin{vmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \cdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{vmatrix}$$

is defined as the Jacobian of $f_1, f_2, \ldots, f_n$ with respect to the n variables $x_1, x_2, \ldots, x_n$ and isdenoted by

$$\frac{\partial(f_1, f_2, \ldots, f_n)}{\partial(x_1, x_2, \ldots, x_n)}$$

If u and v are functions of x and y, then $J(u, v) = \frac{\partial(u,v)}{\partial(x,y)} = \begin{bmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{bmatrix}$

If u, v and w are functions of x, y and z, then $J(u, v, w) = \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix}$

Properties of the Jacobian

1. Chain Rule for Jacobians: If u and v are functions of independent variables r and s andeach of r and s are functions of the variables x and y, then u and v are functions of x and$y$. Further the jacobians satisfy the chain rule$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(r,s)} \cdot \frac{\partial(r,s)}{\partial(x,y)}$

2. If u and v are functions of x and y, then x and y can be solved in terms of u and v. Then
$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{\partial(x, y)}{\partial(u, v)} = 1$$

3. If u, v and w are functions of x, y and z and if u, v, w are functionally related ordependent then
$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = 0$$

SOLVED EXAMPLE

Problems 1. If $u = xyz$, $v = x^2 + y^2 + z^2$, $w = x + y + z$, find $J(x, y, z)$

Solution: $J(x,y,z) = \frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} yz & xz & xy \\ 2x & 2y & 2z \\ 1 & 1 & 1 \end{vmatrix} = -2(x-y)(y-z)(z-x)$$

Therefore

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \frac{-1}{2(x-y)(y-z)(z-x)}$$

Problems 2: If $u = x + y + z$, $u^2v = y + z$, $u^3w = z$, show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = u^{-5}$

Solution: $u_x = u_y = u_z = 1$

$v_x = \frac{-2(y+z)}{u^3}$, $v_y = \frac{u-2(y+z)}{u^3}$, $v_z = \frac{u-2(y+z)}{u^3}$

$w_x = \frac{-3z}{u^4}$, $w_y = \frac{-3z}{u^4}$, $w_z = \frac{1}{u^3} - \frac{3z}{u^4}$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 1 & 1 & 1 \\ \frac{-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} & \frac{u-2(y+z)}{u^3} \\ \frac{-3z}{u^4} & \frac{-3z}{u^4} & \frac{u-3z}{u^4} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & 0 \\ \frac{-2(y+z)}{u^3} & \frac{1}{u^2} & \frac{1}{u^2} \\ \frac{-3z}{u^4} & 0 & \frac{1}{u^3} \end{vmatrix} c_2 \to c_2 - c_1, c_3 \to c_3 - c_1$$

$$= \frac{1}{u^5}$$

Problems 3: If $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1} x + \tan^{-1} y$, show that $\frac{\partial(u,v)}{\partial(x,y)} = 0$

Solution: Let $x = \tan\theta$ and $y = \tan\varphi$,

Then $u = \frac{\tan\theta + \tan\varphi}{1 - \tan\theta\tan\varphi} = \tan(\theta + \varphi)$

And $v = \theta + \varphi$

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(\theta,\varphi)} \cdot \frac{\partial(\theta,\varphi)}{\partial(x,y)}$$

$u_\theta = \sec^2(\theta,\varphi)$, $u_\varphi = \sec^2(\theta,\varphi)$, $v_\theta = v_\varphi = 1$

$$\frac{\partial(u,v)}{\partial(\theta,\varphi)} = \begin{vmatrix} \sec^2(\theta,\varphi) & \sec^2(\theta,\varphi) \\ 1 & 1 \end{vmatrix} = 0$$

Thus, $\frac{\partial(u,v)}{\partial(x,y)} = 0$

Problems 4: Show that $u = x^3 - y^3z^3$, $v = x^2 + y^2z^2 + xyz$ and $w = x - yz$ are functionally dependent andalso find the relation.

Solution: If u, v, w are functionally dependent then $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 0$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 3x^2 & -3y^2z^3 & -3y^3z^2 \\ 2x+yz & 2yz^2+xz & 2y^2z+xy \\ 1 & -z & -y \end{vmatrix}$$

$= yz \begin{vmatrix} 3x^2 & -3y^2z^2 & -3y^2z^2 \\ 2x+yz & 2yz+x & 2yz+x \\ 1 & -1 & -1 \end{vmatrix}$ taking$z$ common from c_2 and y from c_3

$= 0$(Two columns are identical)

Since $x^3 - y^3z^3 = (x-yz)(x^2+xyz+y^2z^2)$

$u = v\,w$ is the relation between the three variables.

1. What is the order of the partial differential equation?

$$\frac{\partial^2 z}{\partial\, y^2} + \left(\frac{\partial z}{\partial y}\right)^2 + \frac{\partial z}{\partial x\,\partial y} + \frac{\partial z}{\partial x} = 0$$

(a) 0 (b) 1

(c) 2 (d) 3

Solution: -(c)

Order of a partial differential equation is the same as that of the order of the highest differential coefficient in it, Therefore, 2 is the order.

2. Form a partial differential equation from

$$x^2 + y^2 + (z-c)^2 = a^2$$

(a) $y\frac{\partial z}{\partial x} - x\frac{dz}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} + x\frac{dz}{\partial y} = 0$

(c) $x\frac{\partial z}{\partial x} - y\frac{dz}{\partial y} = 0$ (d) $x\frac{\partial z}{\partial x} + y\frac{dz}{\partial y} = 0$

Solution: - (a)

$x^2 + y^2 + (z-c)^2 = a^2 \rightarrow$ (1)

Equation (1) contains two arbitrary constants, a and c. Differentiating (1) partially w.r.t x, we get

$$2x + 2(z-c)\frac{\partial z}{\partial\, x} = 0$$

$\Longrightarrow x + (z-c)\frac{dz}{\partial x} = 0$ $\rightarrow$ (2)

Differentiating (1) partially w.r.t y, we get:

$$2y + 2(z-c)\frac{\partial z}{\partial y} = 0$$

$\Rightarrow y + (z-c)\frac{dz}{\partial y} = 0$

Let us eliminate c from (2) & (3)

From (2), $(z-c) = -\frac{x}{\left(\frac{\partial z}{\partial x}\right)}$

Putting this value of $(z-c)$ in (3) we get

$$y - \frac{x}{\left(\frac{\partial z}{\partial x}\right)}\frac{\partial z}{\partial y} = 0$$

$\Rightarrow y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$

3. Form the partial differential equation from:

$$z = f(x^2 - y^2)$$

(a) $y\frac{\partial z}{\partial x} - x\frac{\partial z}{\partial y} = 0$ (b) $y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$

(c) $x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} = 0$ (d) $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$

Solution: - (b)

$z = f(x^2 - y^2)$ → (1)

Differentiating (1) w.r.t x & y:

$\frac{\partial z}{\partial x} = f'(x^2 - y^2)2x$ → (2)

$\frac{\partial z}{\partial y} = f'(x^2 - y^2)(-2y)$ → (3)

Dividing (2) by (3) we get:

$$\frac{\left(\frac{\partial z}{\partial x}\right)}{\left(\frac{\partial z}{\partial y}\right)} = \frac{-x}{y} \quad \Rightarrow y\frac{\partial z}{\partial x} + x\frac{\partial z}{\partial y} = 0$$

4. Solve: $\frac{\partial^3 z}{\partial\, x^2 \partial y} = \cos(2x + 3y)$

(a) $\frac{1}{12}\sin(2x + 3y) + y\,\phi_1(y) + \phi_2\,(y)$

(b) $-\frac{1}{12}\sin(2x + 3y) + x\phi_1(x) + \phi_2(x)$

(c) $-\frac{1}{12}\sin(2x + 3y) + x\,\phi_1\,(y) + \phi_2\,(y)$

(d) $\frac{1}{12}\cos(2x - 3y) + y\,\phi_1(x) + \phi_2\,(x)$

Solution: - (c)

Integrating w.r.t x, we get

$$\frac{\partial^2 z}{\partial x\,\partial y} = \frac{1}{2}\sin(2x + 3y) + f\,(y)$$

Integrating w. r. t x, we get:

$$\frac{\partial z}{\partial y} = -\frac{1}{4}\cos(2x + 3y) + x\int f\,(y)dx + g(y)$$

$$= -\frac{1}{4}\cos(2x + 3y) + x\phi\,(y) + g(y)$$

Integrating w. r. t y, we get:

$$z = -\frac{1}{12}\sin(2x + 3y) + x\int \phi(y)\,dy + \int g\,(y)dy$$

$$= -\frac{1}{12}\sin(2x + 3y) + x\,\phi_1(y) + \phi_2(y)$$

5. Solve: $\frac{\partial^2 z}{\partial x \partial y} = x^2 y$

Subject to the condition z(x,0) = x^2 and z(1,y) = cos y

(a) $\frac{3}{2}x^2y^3 + \cos y + \frac{1}{6}x^2 - 1 + y^2$

(b) $\frac{3}{2}x^2y^3 + \sin y - \frac{1}{6}x^2 + 1 - y^2$

(c) $\frac{1}{6}x^3y^2 + \cos y + \frac{1}{6}y^2 - 1 - x^2$

(d) $\frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2$

Solution:- (d)

$$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = x^2 y$$

On integrating w.r.t x, we obtain

$$\frac{\partial z}{\partial y} = \frac{x^3}{3}y + f(y)$$

Integrating w. r. t y, we obtain

$z = \frac{x^3}{2}.\frac{y^2}{2} + \int f(y)dy + g(x)$ $\qquad [F(y) = \int f(y)dy]$

$\Longrightarrow z = \frac{x^3y^2}{6} + F(y) + g(x)$

Condition 1: Putting $z = x^2 \& y = 0$ in (1), we get

$$x^2 = 0 + F(0) + g(x)$$

Putting the value of $g(x)$ in (1), we get

$z = \frac{x^3y^2}{6} + F(y) + x^2 - F(0)$ → (2)

Condition 2: $z(1, y) = \cos y$

Putting $x = 1$ and $z = \cos y$ in (2), we get

$$\cos y = \frac{y^2}{6} + F(y) + 1 - F(0)$$

Putting the value of $F(y)$ in (2), we obtain

$$z = \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + F(0) + x^2 - F(0)$$

$$= \frac{1}{6}x^3y^2 + \cos y - \frac{1}{6}y^2 - 1 + x^2$$

6. Solve $\frac{\partial^2 z}{\partial y^2} = z$, if $y = 0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

(a) $z = e^{-x} \sin hy + e^x \cos h\, y$

(b) $z = e^{-x} \sin hy - e^x \cos h\, y$

(c) $z = e^x \sin h\, y + e^{-x} \cos h\, y$

(d) $z = (e^x + e^{-x}) \sin h\, y$

Solution:- (a)

If z is a function of y alone, then

$z = A \sin h\, y + B \cos h\, y$,

Where A and B are constants, since Z is a function of $x$and$y$, A and B are the functions of x

Then,

$z = \sin h\, y.f(x) + \cos h\, y.\ \phi\ (x)$ → (1)

On putting $y = 0$ and $z = e^x$ in (1), we obtain

$$e^x = \phi\ (x)$$

(1) Becomes $z = \sin h\, y.f(x) + \cosh y.e^x$ → (2)

On differentiating (2) w. r. t y, we get

$\frac{\partial z}{\partial y} = \cosh y.f(x) + \sin h\, y.e^x$ → (3)

On putting $y = 0$ and $\frac{\partial z}{\partial y} = e^{-x}$ in (3), we obtain

$$e^{-x} = f(x)$$

(2) Becomes $z = e^{-x} \sin h\, y + e^x \cos h\, y$

7. Using method of separation of variables, solve

$$\frac{\partial u}{\partial x} = 2\frac{\partial u}{dt} + u$$

Where $u\ (x, 0) = 6e^{-3x}$

(a) $u = 6e^{-3x-2t}$ (b) $u = 4e^{-2x-t}$

(c) $u = 6e^{2x-3t}$ (d) $u = 6e^{3x+2t}$

Solution: - (a)

$\frac{\partial u}{\partial x} = 2\frac{\partial u}{dt} + u$ → (1)

Let $u = X(x).T(t)$ → (2)

where x is a function of x only & T is a function of t only.

$\therefore\ \frac{\partial (X.T)}{\partial x} = 2\ \frac{\partial}{\partial t}(X.T) + X.T$

$\Longrightarrow T\ \frac{dX}{dx} = 2 \times \frac{dT}{dt} + X.T$

$\Longrightarrow T.X' = 2X.T' + X.T$

$\Longrightarrow T\ \frac{X'}{X} = 2\frac{T'}{T} + 1 = c$ (say)

(a) $\frac{X'}{X} = c \quad \Longrightarrow \frac{1}{X}\frac{dX}{dx} = c \qquad \Longrightarrow \frac{dx}{x} = cdx$

$\Longrightarrow ln\,X = cx + ln(a) \qquad \Rightarrow x = ae^{cx}$

(b) $\frac{2T'}{T} + 1 = c \qquad \Longrightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{2}(c-1)$

$\Longrightarrow ln\,T = \frac{1}{2}(c-1)t + ln\,b \qquad \Longrightarrow T = be^{\frac{1}{2}(c-1)t}$

Putting value of X & T in (2) we get:

$$u = ae^{cx}.be^{\frac{1}{2}(c-1)t}$$

$\Longrightarrow u = abe^{cx+\frac{1}{2}(c-1)t}$ → (3)

$\Longrightarrow u\,(x,0) = abe^{cx}$

But $u(x,0) = 6e^{-3x}$

i.e., $abe^{cx} = 6e^{-3x} \quad \Longrightarrow ab = 6\ \&c = -3$

Putting the values if a b & c in (3), we have

$$u = 6e^{-3x+\frac{1}{2}(-3-1)t}$$

$\Longrightarrow u = 6e^{-3x-2t}$

8. Solve the equation

$$\frac{d^2v}{dx^2} = \frac{\partial v}{dt}$$

Given that $v = 0$ when $t \to \infty$ as well as $v = 0$ at $x = 0\ \&\ x = l$.

(a) $v = constant.\,e^{\frac{n^2\pi^2t^2}{l^2}} \sin\left(\frac{n\pi x}{l}\right)$

(b) $v = constant.\,e^{-\frac{n^2\pi^2t}{l^2}} \sin\left(\frac{n\,\pi\,x}{l}\right)$

(c) $v = constant.\,e^{\frac{-n^2\pi^2t^2}{l^2}} \cos\left(\frac{n\,\pi\,x}{l}\right)$

(d) $v = constant.\,e^{\frac{n^2\pi^2t}{l^2}} \cos\left(\frac{n\,\pi\,x}{l}\right)$

Solution: - (b)

Let $v = X\,T$

$\Longrightarrow \frac{\partial v}{\partial t} = X\ \frac{dT}{dt} \& \frac{\partial^2 v}{\partial x^2} = T\frac{\partial^2 X}{dx^2}$

$\therefore X\frac{dT}{dt} = T\frac{d^2X}{dx^2} = (-p^2)(let)$

$\Longrightarrow \frac{1}{T}\frac{dT}{dt} = \frac{1}{X}\frac{d^2X}{dx^2} = -p^2$ → (2)

$\Rightarrow \frac{1}{T}\frac{dT}{dt} = -p^2$ or $\frac{dT}{dt} + p^2T = 0$ → (3)

And $\frac{1}{X}\frac{d^2X}{dx^2} = -p^2$ or $\frac{d^2X}{dx^2} + p^2X = 0$ → (4)

Solving (3) & (4) we have

$$T = C_1e^{-p^2t}$$

And $X = C_2\cos px + C_3\sin px$

$So,\ v = C_1e^{-p^2t}(C_2\cos px + C_3\sin px)$ → (5)

Putting$x = 0$, $v = 0$ in (5), we get

$O = C_1e^{-p^2t}C_2$

$\therefore\ C_2 = 0$ Since $C_1 \neq 0$ (For non –trivial solution)

On putting the value of C_2 in (5), we get

$v = C_1 e^{-p^2 t} C_3 \sin px$ → (6)

Again putting $x = l$, $v = 0$ in (6) we get

$0 = C_1 e^{-p^2 t}.C_3 \sin pl$, since C_3 cannot be zero

$\therefore \sin pl = 0 = \sin n\,\pi$ $\therefore p = \frac{n\pi}{l}$, n is any integer

On putting the value of p in (6) it becomes

Hence $v = C_1 C_3 e^{-\frac{n^2\pi^2 t}{l^2}} \sin\left(\frac{n\,\pi\,x}{l}\right)$

$$v = constant.\, e^{-\frac{n^2\pi^2 t}{l^2}} \sin\left(\frac{n\,\pi\,x}{l}\right)$$

9. f$x^x.y^y.z^z = c$, find the value of $\frac{\partial^2 z}{\partial x \partial y}$ at $x = y = z$

(a) $(x \log ex)^{-2}$ (b) $-(x \log ex\,)^2$

(c) $-(x \log ex\,)^{-2}$ (d) $-(x \log ex)^{-1}$

Solution: - (d)

Taking log of both sides, we have

$$x \log x + y \log y + z \log z = \log c$$

Differentiating partially w.r.t x, we get

$$x.\frac{1}{x} + \log x + \left(z.\frac{1}{z} + \log z\right)\frac{\partial z}{\partial x} = 0$$

$\Longrightarrow \frac{\partial z}{\partial x} = -\left(\frac{1+\log x}{1+\log z}\right)$

Similarly, $\frac{\partial z}{\partial y} = -\left(\frac{1+\log y}{1+\log z}\right)$

$\Rightarrow \frac{\partial^2 z}{\partial x \partial y} = \frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}\frac{\partial z}{\partial y}$

$= -\frac{(1+\log x)}{(1+\log z)^2}.\frac{1}{z}.\frac{(1+\log y)}{(1+\log z)} = -\frac{(1+\log x)^2}{(1+\log x)^3}.\frac{1}{x}$

$(Since\ x = y = z)$

$= -\frac{1}{x(1+\log x)}$

$= -\frac{1}{x \log e\, x}$

$= -(x \log ex)^{-1}$

10. If $z = x^2 tan^{-1}\left(\frac{y}{x}\right) - y^2 tan^{-1}\left(\frac{x}{y}\right)$, find the value of $\frac{\partial^2 z}{\partial y\, \partial x}$

(a) $\frac{x^2+y^2}{x^2-y^2}$ (b) $\frac{x^2-y^2}{x^2+y^2}$

(c) $\frac{2xy}{x^2+y^2}$ (d) $\frac{2xy}{x^2-y^2}$

Solution: - (b)

We have, $\frac{\partial z}{\partial x} = 2x\ tan^{-1}\left(\frac{y}{x}\right) + \frac{x^2}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) - \frac{y^2}{1+\left(\frac{x}{y}\right)^2}\left(\frac{1}{y}\right)$

$= 2x\ tan^{-1}\left(\frac{y}{x}\right) - \frac{x^2 y}{x^2+y^2} - \frac{y^3}{x^2+y^2}$

$= 2x\ tan^{-1}\left(\frac{y}{x}\right) - \frac{y\,(x^2+y^2)}{(x^2+y^2)} = 2xtan^{-1}\left(\frac{y}{x}\right) - y$

Now,

$$\frac{\partial^2 z}{\partial y\, \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial}{\partial y}\left\{2x\ tan^{-1}\left(\frac{y}{x}\right) - y\right\}$$

$$= \frac{2x}{1+\left(\frac{y}{x}\right)^2}\left(\frac{1}{x}\right) - 1 = \frac{2x^2}{x^2+y^2} - 1 = \frac{x^2-y^2}{x^2+y^2}$$

11. In polar coordinates $x = r\cos\theta \& y = r\sin\theta$, then find $J\left(\frac{\partial(x,y)}{\partial(r,\theta)}\right)$

(a) 2r (b) -2r

(c) r (d) –r

Solution: - (c)

$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial y}{\partial r} = \sin\theta,\ \frac{\partial x}{\partial \theta} = -r\sin\theta, \frac{\partial y}{\partial \theta} = r\cos\theta$

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r$$

12. If $x = u(1-v), y = uv$, then $J\left(\frac{\partial(x,y)}{\partial(u,v)}\right)$ is given by:

(a) u (b) v

(c) $-u(1-2v)$ (d) $u(1-2v)$

Solution: - (a)

$$J\left(\frac{\partial(x,y)}{\partial(u,v)}\right) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$= \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$

$= (u - uv) - (uv)$

$= u$

Previous Year Questions

[JAM 2021]

1. Let (x, y)denote the coordinates in a rectangular Cartesian coordinate system C. Let (x', y')denote the coordinates in another coordinate system C'. Defined by

$$x' = 2x + 3y$$
$$y' = -3x + 4y$$

The area element in C', is

(a) $\frac{1}{17}dx'dy'$ (b) $12dx'dy'$

(c) $dx'dy'$ (d) $x'dx'dy'$

Solution: (a)

The transformation $C \to C'$ is given by,

$$\left.\begin{aligned} x' &= 2x + 3y \\ y' &= 3x + 4y \end{aligned}\right\}$$

Supposes, the area element in C coordinate system is $dx\,dy = dA$ then the area element in C coordinate system can be expressed by $dA' = J(\frac{x,y}{x',y'})\,dx'\,dy'$.

So, $dx\,dy = J\left(\frac{x,y}{x',y'}\right)dx'dy' = \frac{1}{J\left(\frac{x',y'}{x,y}\right)}dx'dy'$

Where $J\left(\frac{x',y'}{x,y}\right)$ is the jacobian of (x', y') w.r.t (x, y). We will prefer to solve it, since we know that $J\left(\frac{x,y}{x',y'}\right) = \frac{1}{J\left(\frac{x',y'}{x,y}\right)}$ and the given transformation is (x', y') in terms of (x, y)

Where, $J\left(\frac{x',y'}{x,y}\right) = \begin{vmatrix} \partial x'/\partial x & \partial x'/\partial y \\ \partial y'/\partial x & \partial y'/\partial y \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ -3 & 4 \end{vmatrix} = 8 - (-9) = 17$

Therefore, the are element in C' coordinate system is $\frac{1}{17}\ dx'dy'$

Correct option is (a)

[JAM 2019]

2. Consider the transformation to a new set of coordinates $(\xi, \eta)$from rectangular Cartesian coordinates(x, y), where $\xi = 2x + 3y$ and $\eta = 3x - 2y$. In the (ξ, η) coordinate system, the area element dxdy is

(a) $\frac{1}{13}d\xi d\eta$ (b) $\frac{2}{13}d\xi d\eta$

(c) $5d\xi d\eta$ (d) $\frac{3}{5}d\xi d\eta$

Solution: **(a)**

Jacobian is

$$J\left(\frac{\boldsymbol{\xi},\boldsymbol{\eta}}{x,y}\right) or\ J(\boldsymbol{\xi},\boldsymbol{\eta}) = \begin{vmatrix} \frac{\partial\xi}{\partial x} & \frac{\partial\xi}{\partial y} \\ \frac{\partial\eta}{\partial x} & \frac{\partial\eta}{\partial y} \end{vmatrix} = \begin{vmatrix} 2 & 3 \\ 3 & -2 \end{vmatrix} = -13$$

We will prefer to solve it, since we know that $J\left(\frac{\xi,\eta}{x,y}\right) = \frac{1}{J\left(\frac{x,y}{\xi,\eta}\right)}$ and the given transformation is $(\boldsymbol{\xi}, \boldsymbol{\eta})$ in terms of (x, y)

Therefore, $J\left(\frac{x,y}{\xi,\boldsymbol{\eta}}\right) = -\frac{1}{13} = J$

Since area element in $\xi - \eta$ system is,

$$dA = \left|J\left(\frac{x,y}{\boldsymbol{\xi},\boldsymbol{\eta}}\right)\right| d\xi d\eta$$

$= \frac{1}{13}d\xi d\eta$

Practice Questions

1. The vibrations of an elastic string is governed by partial differential equation

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 u}{\partial x^2}$$

The length of the string in π and the ends are fixed.

The initial velocity is zero and the initial deflection is $u(x, 0) = 2(sinx + \sin 3x)$, find the deflection $u(x, t)$ of the vibration string for $t > 0$

(a) $u(x, t) = 4 \sin x \sin 2x \sin 2t$

(b) $u(x, t) = 4 \cos x \cos 2x \cos 2t$

(c) $u(x, t) = 4 \cos x \sin 2x \cos 2\ t$

(d) $u\ (x, t) = 4 \sin x \cos 2x \sin 2\ t$

2. A tightly stretched string with fixed end points $x = 0$ and $x = l$ is initially in a position given by$y = y_0 sin^3\left(\frac{\pi x}{l}\right)$. If it is released from rest from this position, find the displacement $y(x, t)$.

(a) $y = \frac{y_0}{4}\left(3 \sin\frac{\pi x}{l}\cos\frac{c\pi t}{l} - \sin\frac{3\ \pi x}{l}\cos\frac{3\ c\ \pi\ t}{l}\right)$

(b) $y = \frac{y_0}{4}\left(2 \cos\frac{\pi\ x}{l}\cos\frac{c\ \pi\ t}{l} - \sin\frac{3\pi x}{l}\sin\frac{3\ c\pi t}{l}\right)$

(c) $y = \frac{y_0}{3}\left(3 \sin\frac{\pi\ x}{l}\sin\frac{c\pi t}{l} - \sin\frac{3\pi x}{l}\sin\frac{3c\pi t}{l}\right)$

(d) $y = \frac{y_0}{4}\left(3 \cos\frac{\pi\ x}{l}\sin\frac{c\pi t}{l} - \cos\frac{3\pi x}{l}\sin\frac{3c\pi t}{l}\right)$

3. Solve the wave equation

$\frac{\partial^2 u}{\partial t^2} = a^2\frac{\partial^2 u}{\partial x^2}$

Under the condition

$u = 0$ When $x = 0$ and $x = \pi$

$\frac{\partial u}{\partial t} = 0$ When $t = 0$ and $u(x, 0) = x,\ 0 < x < \pi$

(a) $u(x,t) = -2\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \sin n\, x \cos nat$

(b) $u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n} \cos n\, x \cos nat$

(c) $u(x,t) = -\sum_{n=1}^{\infty} \frac{1}{n} \cos n\, x \sin nat$

(d) $u(x,t) = -2\ \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \cos n\, x \cos nat$

4. Find $\frac{dz}{dt}$ when$z = xy^2 + x^2y,\ x = at^2,\ y = 2at$

(a) $8\ a^3t^3 + 8\ a^3t^4 + 2a^4t^3$

(b) $16a^3t^3 + 8\ a^3t^4 + 2a^4t^3$

(c) $8a^3t^3 + 16a^3t^4 + 2a^4t^3$

(d) $8a^3t^3 + 4a^3t^4 + 8a^4t^3$

5. Let us define $u = 4x^2 - y^2, x = e^t \cos t \& y = 2e^t \sin t$, find $\frac{du}{dt}$.

(a) $8(\cos 2t\ e^{2t} + \sin 2t\ e^{-2t})$

(b) $8\ e^{-2t}(\cos 2t + \sin 2t)$

(c) $8e^{-2t}(\cos 2t - \sin 2t)$

(d) $8\ e^{2t}(\cos 2t - \sin 2t)$

6. If x increases at the rate of 2 cm/sec at the instant when $x = 3\ cm$ and $y = 1$ cm. At what rate must y be changing in order that the function $2xy - 3x^2y$ shall be neither increasing nor decreasing?

(a) Increasing at the rate of $\frac{32}{21}\ cm/s$

(b) Decreasing at the rate of $\frac{32}{21}\ cm/s$

(c) Increasing at the rate of $\frac{21}{32}\ cm/s$

(d) Decreasing at the rate of $\frac{21}{32}\ cm/s$

7. In polar coordinates $x = r\cos\theta \& y = r\sin\theta$, then find $J\left(\frac{\partial\,(x,y)}{\partial\,(r,\theta)}\right)$

(a) 2r (b) -2r

(c) r (d) -r

8. Define$u = x^2 - y^2,\ v = 2xy,\ x = r\cos\theta,\ y = r\sin\theta$

Find $\frac{\partial\,(u,v)}{\partial\,(r,\theta)}$

(a) $4r^2$ (b) $4r$

(c) $\frac{4}{r}$ (d) $4r^3$

9. A rod of length l with insulated sides is initially at a uniform temperature u. Its ends are suddenly cooled to $0°C$ and are kept at that temperature. The temperature function $u(x_1 t)$ is given by:

(a) $u = \sum_{n=0}^{\infty} b_n \exp\left(\frac{n^2\pi^2c^2}{l^2}t\right)\cos\left(\frac{n\,\pi\,x}{l}\right)$

(b) $u = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2c^2}{l^2}t\right)\sin\left(\frac{n\,\pi\,x}{l}\right)$

(c) $u = \sum_{n=1}^{\infty} b_n \exp\left(-\frac{n^2\pi^2c^2}{l^2}t\right)\sin\left(\frac{2n\,\pi\,x}{l}\right)$

(d) $u = \sum_{n=0}^{\infty} b_n \exp\left(-\frac{n^2\pi^2c^2}{l^2}t\right)\cos\left(\frac{n\,\pi\,x}{l}\right)$

10. Find the solution of $\frac{\partial^2 u}{\partial x^2} = h^2 \frac{\partial u}{\partial t}$ for which
$u(0,t) = u(l,t) = 0, u\,(x,0) = \sin\left(\frac{\pi x}{l}\right)$ by method of variables separable.
(a) $u = \sin\left(\frac{\pi x}{L}\right)\exp\left(\frac{-\pi^2 t}{h^2 L^2}\right)$
(b) $u = \sin\left(\frac{-\pi x}{L}\right)\exp\left(\frac{\pi^2 t}{h^2 L^2}\right)$
(c) $u = \cos\left(\frac{\pi x}{L}\right)\exp\left(\frac{\pi^2 t^2}{h^2 L^2}\right)$
(d) $u = \cos\left(\frac{\pi x}{L}\right)\exp\left(\frac{-\pi^2 t^2}{h^2 L^2}\right)$

11. The ends A and B of a rod 20 cm long have the temperatures at $30°C$ and at $80°C$ until steady state prevails. The temperatures of the ends are changed to $40°C$ and $60°C$ respectively. Find the temperature distribution in the rod at time t.
(a) $u = 20 - x + \frac{20}{\pi}\sum\left[(-1)^n \cos\left(\frac{n\pi x}{20}\right)\exp\left\{-\left(\frac{n\pi c}{20}\right)^2 t\right\}\right]$
(b) $u = 20 + x + \frac{20}{\pi}\sum\left[(-1)^n \cos\left(\frac{n\pi x}{20}\right)\exp\left\{-\left(\frac{n\pi c}{20}\right)^2 t\right\}\right]$
(c) $u = 40 + x - \frac{20}{\pi}\sum\left[\left(\frac{2(-1)^n}{n}\right)\sin\left(\frac{n\pi x}{20}\right)\exp\left\{-\left(\frac{n\pi c}{20}\right)^2 t\right\}\right]$
(d) $u = 40 - x - \frac{20}{\pi}\sum\left[\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{20}\right)\exp\left\{-\left(\frac{n\pi c}{20}\right)^2 t\right\}\right]$

12. A function is defined as $f(x,y) = y^x$. Find the value of $\frac{\partial^2 f}{\partial x \partial y}$ at $x = 2\ \&\ y = 1$.
(a) $ln\ 2$ (b) $\frac{1}{ln\ 2}$
(c) 0 (d) 1

13. If $u = x^3 + y^3 + 3xy^2$ and $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$, then the value of n is:
(a) 0 (b) 1
(c) 2 (d) 3

14. For all (x,y), let $f(x,y) = \begin{cases} \frac{x}{|x|}\sqrt{x^2+y^2} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ What will be the value of $\frac{\partial f}{\partial x}(0,0) + \frac{\partial f}{\partial y}(0,0)$?
(a) 0 (b) 1
(c) – 1 (d) 2

Answer Key									
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
c	a	a	b	d	b	c	d	b	a
11	**12**	**13**	**14**						
c	d	d	b						

Chapter 2

Vector Calculus & Algebra

Coordinate Systems and Transformation.

A **coordinate system** is a method for identifying the location of a point

There are three types of coordinate systems:-

i. Cartesian coordinate system
ii. Cylindrical coordinate system
iii. Spherical coordinate system

Cartesian Coordinate System

A point P can be represented as (x, y, z) .The ranges of the coordinate variables x, y, and z are

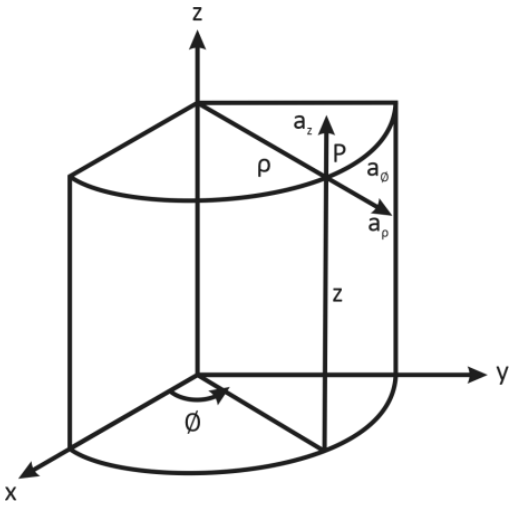
$$-\infty < x < \infty$$
$$-\infty < y < \infty$$
$$-\infty < z < \infty$$

A vector A in Cartesian (otherwise known as rectangular) coordinates can be written as (A_x, A_y, A_z) Or

$$\boldsymbol{A} = A_x a_x + A_y a_y + A_z a_z$$

Where a_x, a_y, and a_z are unit vectors along the x, y and z directions.

Circular Cylindrical Coordinates$(\boldsymbol{\rho}, \boldsymbol{\phi}, \boldsymbol{z})$



Azimuthally angle, is measured from the x-axis in the xy-plane; and z is the same as in the Cartesian system. The ranges of the variables are

$$0 \leq \rho < \infty$$
$$0 \leq \phi < 2\pi$$
$$-\infty < z < \infty$$

A vector **A** in cylindrical coordinates can be written as (A_ρ, A_ϕ, A_z)

$$\boldsymbol{A} = A_\rho a_\rho + A_\phi a_\phi + A_z a_z$$

Where a_ρ, a_θ and a_ϕ are unit vectors along the $r-, \theta -$, andϕ-directions.

The magnitude of A is

$$|A| = \left(A_\rho^2 + A_\theta^2 + A_\theta^2\right)^{\frac{1}{2}}$$

The unit vectors a_ρ, a_θ and a_ϕ are mutually orthogonal; a_r being directed along the radius or in the direction of increasing r, a_θ in the direction of increasingθ, and a_ϕ in the direction of increasingϕ. Thus,

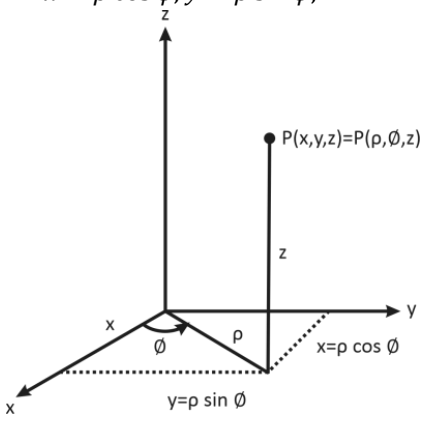
$$a_\rho . a_\rho = a_\theta . a_\theta = a_\phi . a_\phi = 1$$
$$a_\rho . a_\theta = a_\theta . a_\theta = a_\phi . a_\rho = 0$$
$$a_\rho \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_\rho$$
$$a_\phi \times a_\rho = a_\theta$$

The relationship between the variables (x, y, z) of the Cartesian coordinate system and those of the cylindrical system (ρ, ϕ, z) are easily obtained from Figure 2.2 as

$\rho = \sqrt{x^2 + y^2}$, $\phi = \tan^{-1}\frac{y}{x}$, $z = z$

Or

$$x = \rho\cos\phi, y = \rho\sin\phi, z = z$$



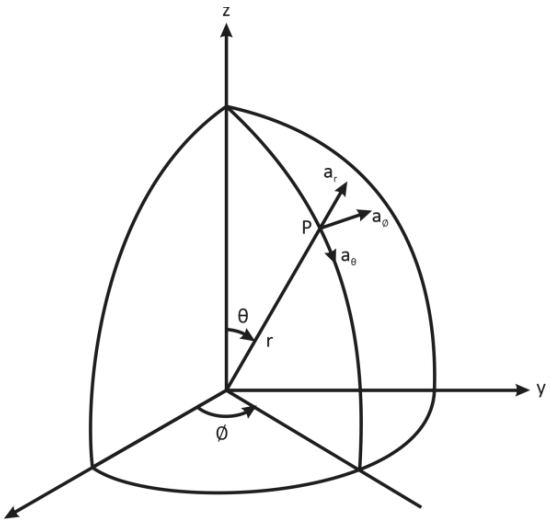
SPHERICAL COORDINATES $(\boldsymbol{r}, \boldsymbol{\theta}, \boldsymbol{\phi})$

A point P can be represented as (r, θ, ϕ)

r is the distance from the origin to point P or the radius of a sphere,

θ is the angle between the z-axis and the position vector of P;

ϕ is measured from the x-axis,



According to these definitions, the ranges of the variables are
$0 \leq r < \infty; 0 \leq \theta \leq \pi; 0 \leq \phi < 2\pi$

A vector A in spherical coordinates may be written as (A_r, A_θ, A_ϕ)

$$A_r a_r + A_\theta a_\theta + A_\phi a_\phi$$

Where a_r, a_θ and a_ϕ are unit vectors along the r, θ, and ϕ directions.
The magnitude of A is

$$|A| = (A_r^2 + A_\theta^2 + A_\theta^2)^{\frac{1}{2}}$$

The unit vectors a_r, a_θ and a_ϕ are mutually orthogonal; a_r being directed along the radius or in the direction of increasing r, a_θ in the direction of increasing θ, and a_ϕ in the direction of increasing ϕ. Thus,

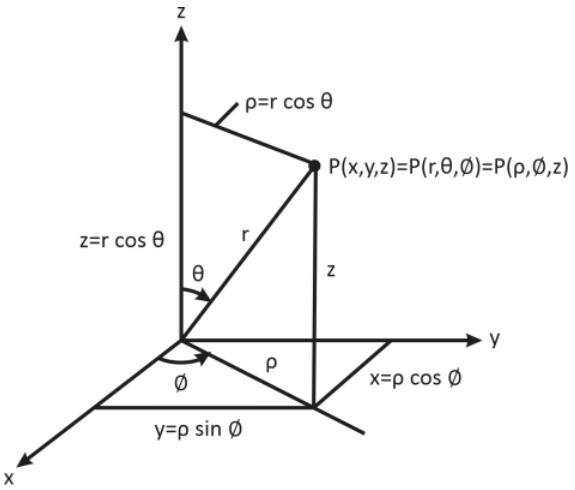
$$a_r . a_r = a_\theta . a_\theta = a_\phi . a_\phi = 1$$
$$a_r . a_\theta = a_\theta . a_\theta = a_\phi . a_r = 0$$
$$a_r \times a_\theta = a_\phi$$
$$a_\theta \times a_\phi = a_r$$
$$a_\phi \times a_r = a_\theta$$

The space variables (x, y, z) in Cartesian coordinates can be related to variables (r, θ, ϕ) of a spherical coordinate system. From Figure 2.5 it is easy to notice that

$$r = \sqrt{x^2 + y^2 + z^2}, \theta = \tan^{-1} \frac{\sqrt{x^2+y^2}}{z}, \phi = \tan^{-1} \frac{y}{x}$$

Or

$$x = r \sin\theta \cos\phi, y = r \sin\theta \sin\phi, z = r \cos\theta$$



Vector Calculus:-
Vector calculus, also known as vector analysis, deals with differentiation and integration of the vector field.

Physical Quantity:-
Those quantities which can be measured by physical Instruments called physical quantity.
E.g. → Mass. Length, temperature, time, etc.

Physical quantities are two types s

- → scalar quantity
- → vector quantity

Scalar Quantity:-
They have only magnitude.
E.g.; mass; time; length, etc.

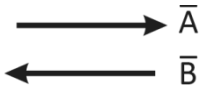
Vector Quantity:- They have magnitude as well as direction.
E.g.; force, momentum, torque.

Some Important Vectors:-

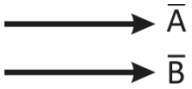
Equal Vectors:- If two vectors have equal magnitude and same direction.
i.e. $\overline{\boldsymbol{A}} = \overline{\boldsymbol{B}}$



Negative Vectors:- if two vectors have equal magnitude but opposite in direction.
i.e. $\overline{\boldsymbol{A}} = -\overline{\boldsymbol{B}}$



Collinear Vectors:- vectors having equal or unequal magnitudes but drawn in the same direction.



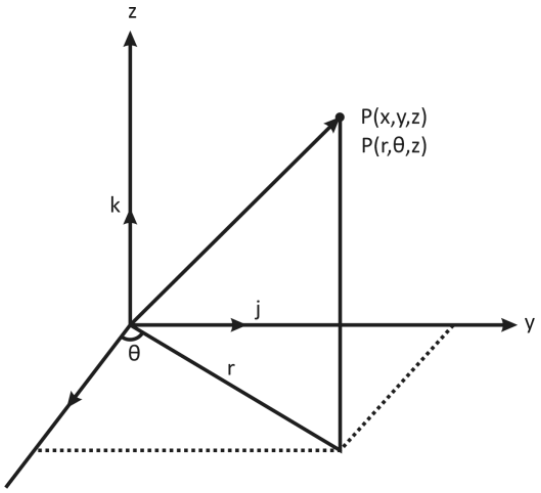
Unit Vector:- A vector of unit magnitude and whose direction in same as that of the given vector is called unit vector.

$$\overline{\boldsymbol{A}} = |\overline{\boldsymbol{A}}|\widehat{\boldsymbol{A}}$$

$\because |\boldsymbol{A}| = \boldsymbol{1}$

$$\widehat{\boldsymbol{A}} = \frac{\overline{\boldsymbol{A}}}{|\boldsymbol{A}|}$$

Vectors expressed in terms of unit vectors



$$\overline{\boldsymbol{A}} = \boldsymbol{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}$$

$$|\overline{\boldsymbol{A}}| = \sqrt{\boldsymbol{x^2 + y^2 + z^2}}$$

$$\widehat{\boldsymbol{A}} = \frac{\overline{\boldsymbol{A}}}{|\overline{\boldsymbol{A}}|}$$

$$\Rightarrow \frac{x\hat{\imath} + y\hat{\jmath} + z\hat{k}}{\sqrt{x^2 + y^2 + z}}$$

Multiplication of Vectors:-

- Scalar Product
- Dot Product
- Cross Product

Scalar Product:- Product of a scalar m to a $\overline{A}$

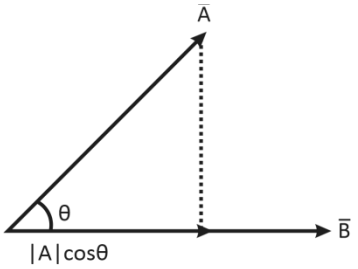
$$\overline{R} = m\overline{A}$$
$$\overline{R} = mA_x\hat{\imath} + m\,A_y\hat{\jmath} + mA_z\hat{k}$$

Dot Product:-

The dot product of two vectors $\overline{A}$ and $\overline{B}$ in expressed with a "dot" between the two vectors as:-

$$\overline{A} \cdot \overline{B} = |A||B| \cos\theta$$

Dot product of two vectors will be a scalar they have only magnitude.



Example:- 1. Find the scalar product of $\overline{A} = 2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}$ and $\overline{B} = 4\hat{\imath} + 2\hat{\jmath} + 3\hat{k}$?

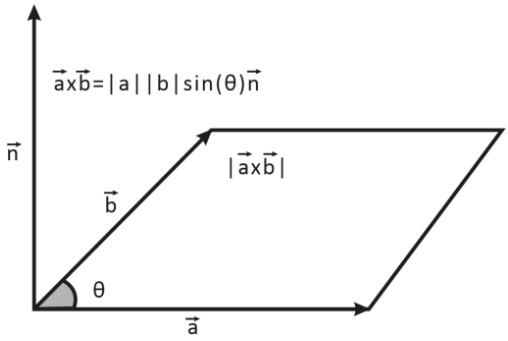
Solution:

$$\overline{A} \cdot \overline{B} = \left(2\hat{\imath} + 3\hat{\jmath} + 2\hat{k}\right) \cdot (4\hat{\imath} + 2\hat{\jmath} + 3\hat{k})$$

$= 8 + 6 + 6 \Rightarrow 20$unit.

Cross Product:- If $\overline{A}$ and $\overline{B}$ are two vectors, then the result of the cross product of these two vectors $\left(\overline{A} \times \overline{B}\right)$ in perpendicular to the both the vectors and normal to plane that contains both the vectors.

$$\overline{A} \times \overline{B} = |A||B| \sin\theta\, \hat{n}$$



- The angle between $\vec{a}$ and $\vec{c}$ is always 90.i.e., $\vec{a}$ and $\vec{c}$ are orthogonal vectors.
- The angle between $\vec{b}$ and $\vec{c}$ is always 90 .i.e., $\vec{b}$ and $\vec{c}$ are orthogonal vectors.
- We can position $\vec{a}$ and $\vec{b}$ parallel to eachother or at an angle of O°, making the resultant vector a zero vector.
- To get the greatest magnitude, the original vectors must be perpendicular (angle of 90°) so that the cross product of the two vectors will be maximum.

Cross Product Properties: -

- The length of the cross product of two vectors :

$$\vec{a} \times \vec{b} = |a||b| \sin(\theta).$$

- Anti-commutative property:

$$\vec{a} \times \vec{b} = -\vec{b} \times \vec{a}$$

- Distributive property:

$$\vec{a} \times (\vec{b} + \vec{c}) = (\vec{a} + \vec{b}) + (\vec{a} \times \vec{c})$$

- Cross product of the zero vector:

$$\vec{a} \times \vec{O} = \vec{O}$$

- Cross product of the vector with itself:

$$\vec{a} \times \vec{a} = \vec{O}$$

- Multiplied by a scalar quantity:

$$\vec{c}(\vec{a} \times \vec{b}) = c\,\vec{a} \times \vec{b} = \vec{a} \times \vec{c}\vec{b}$$

- The cross product of the unit vectors:

$$\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$$

- $i \times j = k;\ j \times k = i;\ \dot{k} \times i = j$

Triple Cross Product

The cross product of a vector with the cross product of the other two vectors is the triple cross product of the vectors. The resultant of the triple cross product is a vector. The resultant of the triple cross vector lies in the plane of the given three vectors.

If **a**, **b**, and **c** are the vectors, then the vector triple product of these vectors will be of the form:

$$(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$$

Note:- $\overline{a} \times (\overline{b} \times \overline{c}) + \overline{b} \times (\overline{c} \times \overline{a}) + \overline{c} \times (\overline{a} \times \overline{b}) = 0$

Example 1:- The expression $\left(\frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}\right)$ is a

(a) Unit vector
(b) Null vector
(c) Vector of magnitude
(d) Scalar

Solution:- Given that,

$$\overline{A} = \frac{1}{\sqrt{2}}\hat{\imath} + \frac{1}{\sqrt{2}}\hat{\jmath}$$

$$|\overline{A}| = \sqrt{\left(\frac{1}{\sqrt{2}}\right)^2 + \left(\frac{1}{\sqrt{2}}\right)^2} \Rightarrow \sqrt{\frac{1}{2} + \frac{1}{2}}$$

$|\overline{A}| = 1$ (unit vector)

Option (a) is correct

Example 2:-

If for two vectors $\overline{a}$ and $\overline{b}$

$$|\overline{a} \times \overline{b}| = |\overline{a} - \overline{b}|$$

Find the angle between $\overline{a}$ and $\overline{b}$

Solution:-

$$|\overline{a} + \overline{b}| = |\overline{a} - \overline{b}|$$

$$|a + b|^2 = |\overline{a} + \overline{b}|^2$$

$$(\overline{a} + \overline{b}).(\overline{a} + \overline{b}) = (\overline{a} - \overline{b}).(\overline{a} - \overline{b})$$

$$\overline{a}^2 + \overline{b}^2 + 2\overline{a}.\overline{b} = \overline{a}^2 + \overline{b}^2 - 2\overline{a}.\overline{b}$$

Or $\overline{a} \cdot \overline{b} = 0$

Thus angle between **a** and **b** is **90**◦.

Example 3:-

If $\overline{a}, \overline{b}, \overline{c}$ be three unit vectors such that $\overline{a} \times (\overline{b} \times \overline{c}) = \frac{1}{2}\overline{b}$, find the angles which $\overline{a}$ makes with $\overline{b}$ and $\overline{c}$ and $\overline{b}$ and $\overline{c}$ being non parallel.

Solution:-

$$\overline{a} \times \overline{(b} \times \overline{c)} = \frac{1}{2}\overline{b}$$

$$(\overline{a}.\overline{c})\overline{b} - (\overline{a}.\overline{b})\overline{c} = \frac{1}{2}\overline{b}$$

$$\overline{a}.\overline{c} = \tfrac{1}{2} \text{ and } \overline{a}.\overline{b} = 0$$

$\Rightarrow \overline{a} \perp \overline{b}$

Now $\overline{a}.\overline{c} = \frac{1}{2}$

$$|a||c| \cos\theta = \frac{1}{2} \Rightarrow \cos\theta = \frac{1}{2}$$

$$\theta = \cos^{-1}\frac{1}{2}; \theta = \frac{\pi}{3}$$

Gradient:- Gradient of a scalar field at a point represent the maximum rate of change of scalar field with respect to position and is always directed along the normal to the field surface at that point & its defined as

$$grad\ \phi = \overline{\nabla}\phi = \left(\frac{\partial}{\partial x}\hat{\imath} + \frac{\partial}{\partial y}\hat{\jmath} + \frac{\partial}{\partial z}\hat{k}\right)\phi$$

$$\overline{\nabla}\phi = \frac{\partial\phi}{\partial x}\hat{\imath} + \frac{\partial\phi}{\partial y}\hat{\jmath} + \frac{\partial\phi}{\partial z}\hat{k}$$

It can be written as –

$$d\phi = \left(\hat{\imath}\frac{\partial\phi}{\partial x} + \hat{\jmath}\frac{\partial\phi}{\partial y} + \hat{k}\frac{\partial\phi}{\partial z}\right).\left(\hat{\imath}dx + \hat{\jmath}dy + \hat{k}dz\right)$$

$$= (\nabla\phi).dr$$

$$d\phi = |\nabla\phi||dr|\cos\theta$$

If $d\overline{r}$ and $\nabla\phi$ are in the same direction,

Then $\theta = 0$ thus $\cos\theta = 1$

$$d\phi = |\nabla\phi||dr|$$

Normal Derivative:-

If $\phi(x, y, z) = k$ respresent a family of surfaces for different values of the constant k.

Then $d\phi = 0$

So, $\nabla\phi.dr = 0$

Thus $\nabla\phi$ is a vector normal to the surface $\phi(x, y, z) = k$.

Directional Derivative:-

If $d\phi \neq 0$

$$\nabla\phi.dr \neq 0$$

Example:-

The temperature at any point in space is given by $T = xy + yz + zx$.

Determine the derivative of T in the direction of the vector $3\hat{\imath} - 4\hat{k}$. At the point (1, 1, 1)

Solution:- Given that

$$T = xy + yz + zx$$

$$\nabla T = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)T$$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)(xy + yz + zx)$

$= \hat{\imath}(y + z) + \hat{\jmath}(x + z) + \hat{k}(x + y)$

Directional derivative at $(1,1,1)$

$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}$

Directional derivative in the direction $(3\hat{\imath} - 4\hat{\jmath})$

$= 2\hat{\imath} + 2\hat{\jmath} + 2\hat{k}.\left(\frac{3\hat{\imath} - 4\hat{\jmath}}{\sqrt{9+16}}\right)$

$= \frac{6-8}{5} \Rightarrow -\frac{2}{5}$

Divergence of a Vector Field:-

The divergence of a vector function $\overline{A}$ is denoted by $div\ \overline{A}$

$$\overline{\nabla}.\overline{A} = \left(\hat{\imath}\frac{\partial A_1}{\partial x} + \frac{\partial A_2}{\partial y} + \frac{\partial A_3}{\partial z}\right)$$

Note:- $div\ \boldsymbol{A}$ is a scalar quantity.

If $u = x^2 + y^2 + z^2$ and $\overline{r} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$, then find $div\ (4\overline{r})$ in term of 4.

$$div\ (u\overline{r}) = \overline{\nabla}.(u\overline{r})$$

$= \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right).[x^2 + y^2 + z^2]\left(x\hat{\imath} + y\hat{\jmath} + z\hat{k}\right)$

$= (3x^2 + y^2 + z^2) + (x^2 + 3y^2 + z^2) + (x^2 + y^2 + 3z^2)$

$= 5x^2 + 5y^2 + 5z^2$

$= 5(x^2 + y^2 + z^2)$

$= 5u$

Curl of a Vector Field:- The curl of a vector point function $\overline{A}$ is defined as known

$$\text{Curl}\ \overline{A} = \overline{\nabla} \times \overline{A}$$

$$\overline{A} = A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}$$

$$\overline{\nabla} \times \overline{A} = \left(\hat{\imath}\frac{\partial}{\partial x} + \hat{\jmath}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \times \left(A_1\hat{\imath} + A_2\hat{\jmath} + A_3\hat{k}\right)$$

$$= \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_1 & A_2 & A_3 \end{vmatrix}$$

$= \hat{\imath}\left(\frac{\partial A_3}{\partial y} - \frac{\partial A_2}{\partial z}\right) - \hat{\jmath}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial z}\right) + \hat{k}\left(\frac{\partial A_2}{\partial x} - \frac{\partial A_1}{\partial y}\right)$

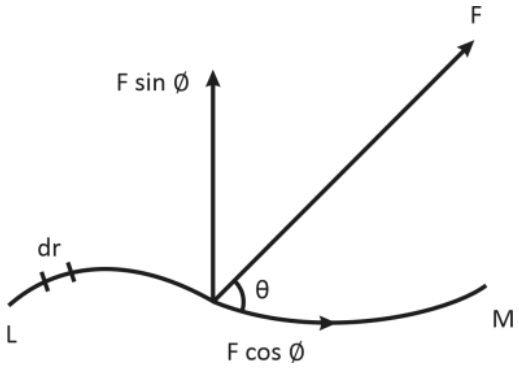
Example:- Calculate the curl of the vector $xyz\,\hat{\imath} + 3x^2y\hat{\jmath} + (xz^2 - y^2z)\hat{k}$

$$curl\ \overline{A} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ xyz & 3x^2y & xz^2 - y^2z \end{vmatrix}$$

$$\overline{\nabla} \times \overline{A} = -2yz\hat{\imath} + (xy - z^2)\hat{\jmath} + (6xy - xz)\hat{k}$$

Line Integral:-

Let $f(xyz)$ be the vector field "The line integral of a vector $\overline{F}$ along the curve LM"



$$I_\ell = \int_L^M (F\cos\theta)dr$$

$$I_\ell = \int_L^M \overline{F}\,.\,\overline{dr}$$

- If $\theta = 0^\circ$, along the curve LM ; $\cos\theta = 1$
  $I_L = \int_L^M \overline{F}\,.\,\overline{dr}$ (Maximum)
- If $\theta = 180^\circ$, opposite the curve LM ; $\cos 180^\circ = -1$
  $I_L = \int_L^M \overline{F}\,.\,d\overline{r}$ (Minimum).
- If $\theta = 90^\circ$, Perpendicular to the curve LM ; $\cos 90^\circ = 0$
  $I_L = \int_L^M \overline{F}\,.\,d\overline{r}$ (Zero).

Example: - Compute the line integral of **F** along a path that runs counter clockwise around the unit circle.

Answer:- We parameterize C by $x = \cos\theta\,, y = \sin\theta$ with $0 \le \theta < 2\pi$.

Then $dx = -\sin\theta\,d\theta$, $dy = \cos\theta\ d\theta$, and:

$\int \boldsymbol{F}.dr = \int_{\theta=0}^{\theta=2\pi} -\sin\theta\,dx + \cos\theta\,dy = \int_0^{2\pi}(\sin^2\theta + \cos^2\theta)\,d\theta = 2\pi.$

Example:- Compute the line integral along the path that goes from (0,0) to (1, 1) by first going along the x-axis to (1, 0) and then going up one unit to (1, 1).

Answer:- To Compute $\int F\ \cdot dr$ we break the curve into two pieces, then add the line integrals along each piece.

First, fix $y = 0$ (so $dy = 0$) and let x range from 0 to 1.

$$\int_{x=0}^{x=1} F\cdot dr = \int_{x=0}^{x=1} -ydx + xdy = \int_0^1 0\,dx = 0.$$

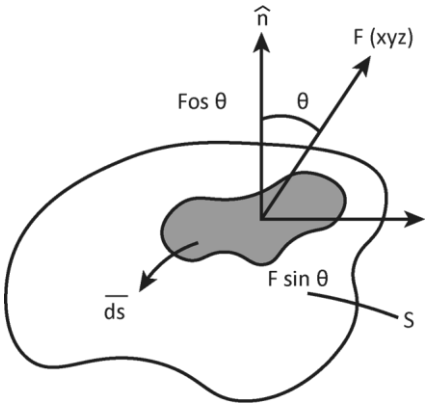
Next, fix $x = 1$ (so $dx = 0$) and let y range from 0 to 1:

$$\int_{y=0}^{y=1} F\cdot dx = \int_{y=0}^{y=1} -ydx + 1\,dy = 1.$$

We conclude that $\int \boldsymbol{F}.dr = 1$.

Surface Integrals:-

Let $F(x, y, z)$ be the vectors



"The integral of the components of $\overline{F}$ along the normal to the surface"

$$I_S = \int (F\cos\theta)dS$$
$$I_S = \int (F \cdot \hat{n})dS$$

Example:- Evaluate $\iint xyz\, dS$,

In surface S which is a part of the plane where $Z = 1 + 2x + 3y$, which lies above the rectangle $[0,3] * [0,2]$.

Solution:-

Given $\iint xyz\, dS$ and $Z = 1 + 2x + 3y$

$$= \int_0^3 \int_0^2 xy\,(1 + 2x + 3y)\sqrt{(2)^2 + (3)^2 + 1}\,dydx$$

$$= \int_0^3 \int_0^2 (\,xy + 2x^2y + 3xy^2)\,\sqrt{14}\,dydx$$

$$= \sqrt{14}\int_0^3 \left[\frac{xy^2}{2} + \frac{2x^2y^2}{2} + \frac{3xy^3}{3}\right]_0^2 dx$$

Now substitute the limits, we get

$$= \sqrt{14}\int_0^3 [2x + 4x^2 + 8x]\,dx$$

Differentiate with respect to x, we get

$$= \sqrt{14}\left[\frac{2x^2}{2} + \frac{4x^3}{3} + \frac{8x^2}{2}\right]_0^3$$

Now substitute the limits,

$$= \sqrt{14}(9 + 36 + 36)$$
$$= 81\sqrt{14}$$

Volume Integral:-

In mathematical, a volume integral in cylindrical coordinates....

$\iiint_v f\,(\rho, \varphi, z)\rho d\rho d\varphi dz,$

And a volume integral in spherical coordinates.....

$\iiint_v f(r, \theta, \varphi)\, r^2 \sin\theta\, dr d\theta d\varphi.$

Example:-

Evaluate the integral

$$\int_0^1 \int_0^x \int_0^{1+x+y} f(x,y,z)\, dz\, dy\, dx$$

Where $f(x,y,z) = 1$.

Solution:-

$$\int_0^1 \int_0^x \int_0^{1+x+y} dz dy dx$$

$$= \int_0^1 \int_0^1 \left(z|_{z=0}^{z=1+x+y}\right) dy\, dx$$

$$= \int_0^1 \int_0^x (1+x+y)\, dy\, dx$$

$$= \int_0^1 \left[y + yx + \frac{y^2}{2}\right]_{y=0}^{y=x} dx$$

$$= \int_0^1 \left(x + x^2 + \frac{x^2}{2}\right) dx$$

$$= \int_0^1 \left(x + \frac{3x^2}{2}\right) dx$$

$$= \left[\frac{x^2}{2} + \frac{x^3}{2}\right]_0^1 = \frac{1}{2} + \frac{1}{2} = 1$$

Stokes' Theorem:-

The Stoke's theorem states that "the surface integral of the curl of a function over a surface bounded by a closed surface is equal to the line integral of the particular vector function around that surface."

$$\oint_C \vec{F}.\vec{dr} = \iint_S (\nabla \times \vec{F}).\vec{dS}$$

Where,

C = A closed curve.

S = Any surface bounded by C.

F = A vector field whose components have continuous derivatives in an open region of R^3 containing S.

Example:-

Verify Stoke's theorem for the vector field vector $F = (2x-y)i - yz^2 j - y^2 z\, k$ over the upper half surface of $x^2 + y^2 + z^2 = 1$ bounded by its projection on the xy-plane.

Using stoke's theorem

$$\int_C \vec{F} \cdot \vec{dr} = \iint_S curl\, \vec{F} \cdot \hat{n} dS$$

C is the circle:- $x^2 + y^2 = 1, z = 0$ (xy-plane)

i.e. $x = \cos t\,, y = \sin t\,, z = 0$

$r = x\hat{\imath} + y\hat{\jmath}$; Where $0 \le \theta < 2\pi$

$\overrightarrow{dr} = dx\vec{\imath} + dy\vec{\jmath}$

Where, $\vec{F} = (2x-y)\vec{\imath} - yz^2\vec{\jmath} - y^2 z\vec{k}$

$\therefore \vec{F} \cdot \vec{dr} = (2x-y)dx;\ (\because z = 0)$

L.H.S $= \int_C \vec{F} \cdot \overrightarrow{dr} = \int_C (2x-y)\, dx$

$$= \int_0^{2\pi} (\sin^2 t - 2\cos t \sin t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t - \sin 2t)\, dt$$

$$= \int_0^{2\pi} (\sin^2 t - \sin 2t)\, dt$$

$$= \int_0^{2\pi} \left\{\frac{1}{2}(1 - \cos 2t) - \sin 2t\right\} dt$$

$$= \left[\frac{t}{2} - \frac{\sin 2t}{4} + \frac{\cos 2t}{2}\right]_0^{2\pi}$$

$$= \left(\frac{1}{2} - \frac{1}{2}\right) + (\pi - 0) = \pi$$

Hence, $\vec{F} \cdot \overrightarrow{dr} = \pi$ .... (1)

Also,

Curl $\vec{F} = \nabla \times \vec{F}$

$$= \begin{vmatrix} \vec{\imath} & \vec{\jmath} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x - y & -yz^2 & -y^2 z \end{vmatrix}$$

$$= \vec{\imath}(-2yz + 2yz) - \vec{\jmath}(0) + \vec{k}(0 + 1)$$

$$= \vec{k}$$

$$\therefore \overrightarrow{dS} = \hat{n} dS$$

$$= dydz\, i + dzdx\, j + dxdy\, k$$

Hence,

R.H.S. $= \iint_S curl\, \vec{F}.\hat{n} dS = \iint dx\, dy$ .... (2)

$$= \pi$$

$\because \iint dx\, dy$ represents the area of the

Circle $x^2 + y^2 = 1$ which is π.

Thus, from (1) and (2) we conclude that the theorem is verified.

Divergence Theorem

The divergence theorem states that the surface integral of the normal component of a vector point function "F" over a closed surface "S" is equal to the volume integral of the divergence of F taken over the volume "V" enclosed by the surface S.

Thus, the divergence theorem is symbolically denoted as:

$$\int_S (\overline{F}.\hat{n})\, dS = \int_v (\overline{\nabla}.\overline{F})\, dV$$

Example → Compute $\iint_S F.dS$ where

$$F = \left(3x + z^{77}, y^2 - \sin x^2 z, xz + ye^{x^5}\right)$$

And S is surface of box

$$0 \le x \le 1, 0 \le y \le 3, 0 \le z \le 2.$$

Use outward normal **n**.

Solution:

The divergence of F is: